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STABILITY FOR FINITE ELEMENT DISCRETIZATION OF SOME INVERSE PARAMETER PROBLEMS FROM INTERNAL DATA - APPLICATION TO ELASTOGRAPHY

ELIE BRETIN*, PIERRE MILLIEN†, AND LAURENT SEPPECHER‡

Abstract. In this article, we provide stability estimates for the finite element discretization of a class of inverse parameter problems of the form $-\nabla \cdot (\mu S) = \boldsymbol{f}$ in a domain Ω of $\mathbb{R}^d$. Here μ is the unknown parameter to recover, the matrix valued function S and the vector valued distribution $\boldsymbol{f}$ are known. As uniqueness is not guaranteed in general for this problem, we prove a Lipschitz-type stability estimate in an hyperplane of $L^2(\Omega)$. This stability is obtained through an adaptation of the so-called discrete *inf-sup* constant or LBB constant to a large class of first-order differential operators. We then provide a simple and original discretization based on hexagonal finite element that satisfies the discrete stability condition and shows corresponding numerical reconstructions. The obtained algebraic inversion method is efficient as it does not require any iterative solving of the forward problem and is very general as it only requires S and μ to be bounded and no additional information at the boundary is needed.

Key words. Inverse problems, Reverse Weak Formulation, Inf-Sup constant, Linear Elastography, Finite Element Method

AMS subject classifications. 65J22, 65N21, 35R30, 65M60

1. Introduction.

This work deals with inverse problems of the form

$$-\nabla \cdot (\mu S) = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^d$, $d \geq 2$ and where $\mu \in L^\infty(\Omega)$ is the unknown parameter map. In this problem, $S \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ and $\boldsymbol{f} \in H^{-1}(\Omega, \mathbb{R}^d)$ are given from some measurements and may contain noise. If one defines the first order differential operator

$$\begin{aligned} T : L^\infty(\Omega) \subset L^2(\Omega) &\to H^{-1}(\Omega, \mathbb{R}^d) \\ \mu &\mapsto -\nabla \cdot (\mu S), \end{aligned} \tag{1.2}$$

the inverse parameter problem that we aim to solve can be expressed as

$$\text{Find } \mu \in L^\infty(\Omega) \quad \text{s.t.} \quad T\mu = \boldsymbol{f}. \tag{1.3}$$

If S, $\boldsymbol{f}$ and μ are assumed smooth enough, this problem reads as a first order transport equation in μ that can be solved with the characteristics method knowing μ in a part of the boundary (the incoming flow boundary). Here, as no additional regularity is assumed and as the right-hand side $\boldsymbol{f}$ belongs to $H^{-1}(\Omega, \mathbb{R}^d)$ this problem shall be considered under its weak formulation :

$$\text{Find } \mu \in L^\infty(\Omega) \quad \text{s.t.} \quad \langle T\mu, \boldsymbol{v}\rangle_{H^{-1},H^1_0} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{H^{-1},H^1_0}, \quad \forall \boldsymbol{v} \in H^1_0(\Omega, \mathbb{R}^d). \tag{1.4}$$

This weak form of the inverse problem (1.3) (introduced in [1]) will be called the Reverse Weak Formulation (RWF). In this inverse problem, we do not assume the

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knowledge of any information on μ at the boundary. Note that the case $\boldsymbol{f} = \boldsymbol{0}$ can be considered and corresponds to the determination of the null space of the operator T.

In [1] the well-posedness of this problem has been established in an hyperplane of $L^2(\Omega)$ in a general setting and under light hypothesis of regularity and invertibility of the matrix S. See subsection 1.1 and [1]. Note that the hypotheses used in this reference will not be used to established the error estimates given in the present paper.

The goal of the present paper is to investigate the stability properties of the discretized version of the problem (1.4) and to provide error estimates based on the properties of the discretization spaces and on the discretized approximation of the operator T. More precisely, given a finite dimensional operator $T_h : M_h \to V_h'$ and $\boldsymbol{f}_h \in V_h'$ where M_h and V_h are finite dimensional subspaces that approach $M := L^2(\Omega)$ and $V := H_0^1(\Omega, \mathbb{R}^d)$ respectively, we seek conditions on M_h, V_h and T_h for the L^2-stability of the following discretized problem:

$$\text{Find } \mu_h \in M_h \quad \text{s.t.} \quad T_h \mu_h = \boldsymbol{f}_h. \tag{1.5}$$

We also give conditions that guarantee the convergence of μ_h to μ for the L^2-norm. In most cases, the stability only occurs in an hyperplane of $L^2(\Omega)$ which is the orthogonal of the singular direction of the operator T_h with respect to its smallest singular value. This leads to a remaining scalar uncertainty that can be resolved using a single additional scalar information on μ.

The originality of this work lies here on the Reverse Weak Formulation (1.4) that exhibits the unknown parameter μ as the solution of a weak linear differential problem in the domain Ω without boundary condition. Hence the uniqueness is not guaranteed at first look and the stability has to be considered with respect to some possible errors on both $\boldsymbol{f}$ and T. As we will see, the error term $T_h - T$ is not controlled in $\mathscr{L}\left(L^2(\Omega), H^{-1}(\Omega, \mathbb{R}^d)\right)$ (definition in Section 2) in general but only for a weaker norm (see Subsection 2.3). This creates difficulties that are not covered by the classical literature on the theory of perturbations of linear operators.

1.1. Scientific context and motivations. Elastography is an imaging modality that aims at reconstructing the mechanical properties of biological tissues. The local values of the elastic parameters can be used as a discriminatory criterion for differentiating healthy tissues from diseased tissues [20]. While numerous modalities of elastography exist (see for example [13, 18, 10, 7]), the most common procedure is to use an auxiliary imaging method (such as ultrasound imaging, magnetic resonance imaging, optical coherence tomography . . .) to measure the displacement field $\boldsymbol{u}$ in a medium when a mechanical perturbation is applied. See [21] and inside references for recent advances on this point. The inverse problem can be formulated as recovering the shear modulus μ in the linear elastic equation

$$-\nabla \cdot (2\mu\mathcal{E}(\boldsymbol{u})) - \nabla(\lambda \nabla \cdot \boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.6}$$

where $\boldsymbol{u}$ and $\boldsymbol{f}$ are given in Ω and λ can be assumed known in Ω. The term $\mathcal{E}(\boldsymbol{u})$ denotes the strain matrix which is the symmetric part of the gradient of $\boldsymbol{u}$. The stability of this inverse problem has been extensively studied under various regularity assumptions for the coefficients to be reconstructed [2, 3, 23, 16]. Recently, in [1] the authors introduced a new inversion method based on a finite element discretization of equation (1.1) where $S := 2\mathcal{E}(\boldsymbol{u})$. A study of the linear operator T defined by (1.2) or by the equivalent weak formulation

$$\langle T\mu, \boldsymbol{v}\rangle_{H^{-1}, H_0^1} := \int_\Omega \mu S : \nabla \boldsymbol{v}, \quad \forall \boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^{d\times d}) \tag{1.7}$$

showed that, under a piecewise smoothness hypothesis on S and under an assumption of the form $|\det(S)| \geq c > 0$ *a.e.* in Ω, the operator T has a null space of dimension one at most and is a closed range operator. This ensures the theoretical stability of the reconstruction in the orthogonal complement of the null space. However, depending on the choice of discretization spaces, the discretized version of T may not satisfy the same properties and numerical instability may be observed. For instance, in [1] the authors approach (1.7) using the classical pair $(\mathbb{P}^0, \mathbb{P}^1)$ of finite element spaces. As it could have been expected, they faced a numerical instability that was successfully overcome by using a TV-penalization technique.

Remark 1.1. The classical elliptic theory says that the strain matrix belongs to $L^2(\Omega, \mathbb{R}^{d\times d})$. Here, we add the hypothesis $S \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ in order to control the error on μ in the Hilbert space $L^2(\Omega)$. This boundedness hypothesis is not restrictive as it is known that the strain is bounded as soon as the elastic parameters are piecewise smooth with smooth surfaces of discontinuity (see [17]).

Let us point out here that inverse problems of the form (1.1) may arise from various other physical situations. Note first that the reconstruction of the Young's modulus E when the Poisson's ratio ν is known is very similar to the problem defined in (1.6). In this case the governing linear elastic equation reads $-\nabla\cdot(E\,\Sigma) = \boldsymbol{f}$ where $\Sigma := a_\nu \mathcal{E}(\boldsymbol{u}) + b_\nu(\nabla\cdot\boldsymbol{u})I$ and $a_\nu := 1/(1+\nu)$ and $b_\nu := \nu/((1+\nu)(1-2\nu)$ in dimension 3. A second example is the electrical impedance imaging with internal data, where the goal is to recover the conductivity σ in the scalar elliptic equation $-\nabla\cdot(\sigma\nabla u) = 0$. If one can measure two potential fields u_1 and u_2 solutions of the previous equation and defines $S := [\nabla u_1 \ \nabla u_2]$, then the problem reads $-\nabla\cdot(\sigma S) = \boldsymbol{0}$. A third example is a classical problem corresponding to the particular case where S is the identity matrix everywhere. In this case, the problem reads $-\nabla\mu = \boldsymbol{f}$ which is the inverse gradient problem.

The properties of the gradient operator $\nabla : L^2(\Omega) \to H^{-1}(\Omega, \mathbb{R}^d)$ and its discretization have been extensively studied in particular in the context of fluid dynamics and some tools developed in this framework are useful to treat our more general problem. For the reader convenience, let us recall the most important property which ensures the existence of a bounded left-inverse.

Hence, in the case where S is the identity matrix everywhere, *i.e.* $T := -\nabla$, the operator T is known to be a closed range operator from $L^2(\Omega)$ to $H^{-1}(\Omega, \mathbb{R}^d)$ if Ω is a Lipschitz domain (see [22, p.99] and references within). One can write

$$\|q\|_{L^2(\Omega)} \leq C\,\|\nabla q\|_{H^{-1}(\Omega)} \quad \forall q \in L^2_0(\Omega),$$

where $C > 0$ and $L^2_0(\Omega)$ is the space of zero-mean, square-integrable functions. The norm of the pseudo-inverse of the gradient in $H^{-1}(\Omega, \mathbb{R}^d)$ is closely related with the *inf-sup* condition of the divergence:

$$\beta := \inf_{q\in L^2_0(\Omega)} \sup_{\boldsymbol{v}\in H^1_0(\Omega,\mathbb{R}^d)} \frac{\int_\Omega (\nabla\cdot\boldsymbol{v})q}{\|\boldsymbol{v}\|_{H^1_0(\Omega)}\,\|q\|_{L^2(\Omega)}} > 0 \qquad (1.8)$$

Indeed, we have $C = 1/\beta$. Since the closed-range property of the gradient is equivalent to the surjectivity of the divergence in $L^2_0(\Omega)$, the study of behavior of β is an important step in establishing the well-posedness and stability of the Stokes problem [14, Chap. I, Theorem 4.1]. The constant β is also known as the *LBB* constant (for Ladyzhenskaya-Babuška-Brezzi). It is well known that in general, the constant β may not behave well in finite element spaces, and may vanish when the mesh size goes to zero. More precisely, if one considers discrete spaces $M_h \subset L^2(\Omega)$ and $V_h \subset H_0^1(\Omega, \mathbb{R}^d)$ with discretization parameter $h > 0$, the associated discrete *inf-sup* constant given by

$$\beta_h := \inf_{\substack{q \in M_h \\ q \perp 1}} \sup_{\boldsymbol{v} \in V_h} \frac{\int_\Omega (\nabla \cdot \boldsymbol{v}) q}{\|\boldsymbol{v}\|_{H_0^1(\Omega)} \|q\|_{L^2(\Omega)}} \tag{1.9}$$

may not satisfy the discrete *inf-sup* condition $\forall h > 0, \beta_h \geq \beta^* > 0$. Pairs of finite element spaces that satisfy the discrete *inf-sup* condition are known as *inf-sup* stable elements and play an important role in the stability of the Galerkin approximation for the Stokes problem. We refer to [5] for more details on the *inf-sup* constant of the gradient and its convergence.

1.2. Main results. Inspired by this approach, we introduce a generalization of the *inf-sup* constant and a corresponding definition of the discrete *inf-sup* constant that are suitable for operators of type (1.2) in particular. A major difference with the classical definition of the *inf-sup* constant of the gradient is that, here, the operator T may contain measurement noise and may have a trivial null space.

In a general framework, consider $T \in \mathscr{L}(M, V')$ where M and V are two Hilbert spaces. The problem $T\mu = \boldsymbol{f}$ is approached by a finite dimensional problem $T_h \mu_h = \boldsymbol{f}_h$ where $T \in \mathscr{L}(M_h, V_h')$ and M_h, V_h approach M and V respectively.

The first main goal of this work is to provide a stability condition with respect to the M-norm for the discrete problem based on the associated discrete *inf-sup* constant. We consider the stability with respect to both the noise and the interpolation error on the right-hand side $\boldsymbol{f}$ and on the operator T itself. The case $\boldsymbol{f} = \mathbf{0}$ corresponds to a null space identification problem and the condition $\|\mu\|_M = 1$ is added.

In Theorem 4.1 we provide an error estimate between the normalized solution of $\arg\min_{z_z \in M_h} \|T_h z_h\|_{V_h'}$ and the normalized solution of $Tz = \mathbf{0}$ of the form

$$\|z_h - p_h(z)\|_M \leq \frac{C}{\beta(T_h)} \big(\|T - T_h\| + \|z - \pi_h z\|_M \big)$$

where $\pi_h z$ is the orthogonal projection of z on M_h, $p_h(z) := \pi_h z / \|\pi_h z\|_M$. The constant $\beta(T_h)$ is an adaptation of the *inf-sup* constant from (1.9) to general operators. (See Section 3). For the hypotheses and other details about the norms used, see directly Theorem 4.1.

In the case $\boldsymbol{f} \neq \mathbf{0}$, we consider two distinct situations. The first case is when T is invertible and $\alpha(T) := \inf_{z \in N} \|Tz\|_{V'} / \|z\|_M$ is not "too small". In Theorem 4.4 we provide an error estimate between the solution of $T_h \mu_h = \boldsymbol{f}_h$ in the sense of least squares and the unique solution of $T\mu = \boldsymbol{f}$ of the form

$$\frac{\|\mu_h - \pi_h \mu\|_M}{\|\pi_h \mu\|_M} \leq \frac{C}{\alpha\,(T_h)} \big(\|T - T_h\| + \|\boldsymbol{f} - \boldsymbol{f}_h\| + \|\pi_h \mu - \mu\|_M \big).$$

where $\alpha(T_h) := \min_{z_h \in M_h} \|T_h z_h\|_{V_h'} / \|z_h\|_M$. For the hypotheses and other details about the norms used, see directly Theorem 4.4.

The second case is when T has a non trivial null space (of dimension one) or remains invertible but with a constant $\alpha(T_h)$ too small to make the previous result applicable. In this case, the error estimate is only proved in an hyperplane of M (the orthogonal complement of the approximated null space). The approximation of μ is then obtained up to an unknown scalar constant.

In Theorem 4.7 we provide an error estimate between the solution of $T_h\mu_h = \boldsymbol{f}_h$ in $\{z_h\}^\perp$ in the sense of least squares and the solution of $T\mu = \boldsymbol{f}$ up to an unknown translation in the direction z_h. This estimate is of the form: $\exists t \in \mathbb{R}$ such that

$$\frac{\|\mu_h + t\, z_h - \pi_h\mu\|_M}{\|\pi_h\mu\|_M} \leq \frac{C}{\beta\,(T_h)}\big(\, \|T_h - T\| + \|\boldsymbol{f}_h - \boldsymbol{f}\| + \|\pi_h\mu - \mu\|_M + \alpha\,(T_h)\,\big).$$

For the hypotheses and other details about the norms used, see directly Theorem 4.7.

These error estimates are quantitative. They depend on the discrete *inf-sup* constant and can be explicitly computed in all practical situations dealing with experimental data. These estimates allow for a control of the quality of the reconstruction in the pair of approximation spaces (M_h, V_h) directly from the noisy interpolated data. The behavior of the discrete *inf-sup* constant with respect to the discretization parameter h gives a practical criterion for the convergence of μ_h towards μ.

The present paper is closely linked to the sensitivity analysis and discretization analysis for the Moore-Penrose generalized inverse of T when T is a closed range operator. There exist a vast literature on this subject (see [4, 9, 24, 15] and references herein) as well as on the finite dimensional interpolation of the generalized inverse [11]. However, there are fundamental differences between the present work and the existing literature. First, we do not know here whether the operator T has closed range. Second, we perform a sensitivity analysis of the left inverse of $T \in \mathscr{L}\,(M, V')$ under perturbations that are controlled in a weaker norm. More precisely, perturbations are controlled here in $\mathscr{L}(E, V')$ where $E \subset M$ is a Banach space dense in M. This might seem a technical issue but it is mandatory if one wants to work with discontinuous parameters μ and S. This choice is motivated by the applications in bio-medical imaging where, in most cases, the biological tissues exhibit discontinuities in their physical properties. For instance, in the linear elasticity inverse problem (see equation (1.6)) the matrix $S = 2\mathcal{E}(\boldsymbol{u})$ has the same surfaces of discontinuities than the shear modulus of the medium and cannot be approached in $L^\infty(\Omega, \mathbb{R}^{d\times d})$ by smooth functions. This leads to perturbations of T in $\mathscr{L}\left(L^\infty(\Omega), H^{-1}(\Omega, \mathbb{R}^d)\right)$ instead of $\mathscr{L}\left(L^2(\Omega), H^{-1}(\Omega, \mathbb{R}^d)\right)$. More details and examples are given in Subsection 2.3.

1.3. Outline of the paper. The article is organized as follows: In Section 2, we describe the Galerkin approximation of the problem (1.3) and define all the approximation errors involved. In Section 3, we generalize the notion of *inf-sup* constant to any operator $T \in \mathscr{L}(M, V')$ and we prove in Theorem 3.10 the upper semi-continuity of the discrete *inf-sup* constant. This is an asymptotic comparison between the discrete and the *continuous inf-sup* constants. In Section 4 we give and prove the main stability estimates (Theorems 4.1, 4.4 and 4.7) based on the discrete version of the *inf-sup* constant just defined. In Section 5 we present various numerical inversions, including stability tests and numerical computations of the *inf-sup* constant for various pairs of finite element spaces. We also introduce in this section a pair of finite element spaces based on a hexagonal tilling of the domain Ω. It shows excellent numerical stability properties when compared to some more classical pair of discretization spaces.

2. Discretization using the Galerkin approach.

We describe the Galerkin approximation of problem (1.3) and give the definitions of the various errors of approximation.

2.1. General notations. In all this work, M and V are two Hilbert spaces with respective inner products denoted $\langle .,. \rangle_M$ and $\langle .,. \rangle_V$. We denote $E \subset M$ a Banach space dense in M. The space $V' := \mathscr{L}(V, \mathbb{R})$ is the space of the bounded linear forms on V endowed with the operator norm

$$\|\boldsymbol{\varphi}\|_{V'} := \sup_{\boldsymbol{v} \in V} \frac{\langle \boldsymbol{\varphi}, \boldsymbol{v} \rangle_{V',V}}{\|\boldsymbol{v}\|_V}, \tag{2.1}$$

where $\langle ., . \rangle_{V',V}$ is the duality pairing between V' and V. The space $\mathscr{L}(M, V')$ is the space of the bounded linear operator from M to V' endowed with the operator norm written $\|.\|_{M,V'}$. For any $T \in \mathscr{L}(M, V')$, we denote its null space by $N(T)$.

Example 2.1. In the case of the inverse elastography problem using the operator T defined in (1.2), we take $M := L^2(\Omega)$, $V := H_0^1(\Omega, \mathbb{R}^d)$, $E := L^\infty(\Omega)$ and so $V' = H^{-1}(\Omega, \mathbb{R}^d)$. Here $H_0^1(\Omega, \mathbb{R}^d)$ is the space of all squared integrable vector-valued functions $\boldsymbol{v}$ on Ω such that $\nabla \boldsymbol{v}$ is also square integrable and such that its trace on $\partial\Omega$ vanishes. The space $H^{-1}(\Omega, \mathbb{R}^d)$ is the topological dual of $H_0^1(\Omega, \mathbb{R}^d)$.

2.2. Spaces discretization and projection. In order to approach the problem (1.3) by a finite dimensional problem, we first approach spaces M and V by finite dimensional spaces.

Definition 2.2. *For any Banach space X, we say that a sequence subspaces $(X_h)_{h>0}$ approaches X if this sequence is asymptotically dense in X. That means that for any $x \in X$, there exists a sequence $(x_h)_{h>0}$ such that $x_h \in X_h$ for all $h > 0$ and $\|x_h - x\|_X$ converges to zero when h goes to zero. We naturally endow X_h with the restriction of the X-norm to make it a normed vector space.*

Consider now two sequences of subspaces $(M_h)_{h>0}$ and $(V_h)_{h>0}$ that approach respectively the Hilbert spaces M and V. Naturally, M_h is endowed with the M-norm and V_h is endowed with the V-norm. In some cases we need to use the E-norm over M_h. To highlight the difference, we will denote $E_h := (M_h, \|.\|_E)$ the space M_h endowed with the E-norm.

Example 2.3. In the case of Example 2.1, $M = L^2(\Omega)$ and one can choose M_h as the classical finite element space $\mathbb{P}^0(\Omega_h)$, *i.e.* the class of piecewise constant functions over a subdivision of Ω by elements of maximum diameter $h > 0$ [14].

Definition 2.4. *We denote $\pi_h : M \to M_h$ the orthogonal projection form M onto M_h. It naturally satisfies $\lim_{h \to 0} \|\pi_h m - m\|_M = 0$ and $\|\pi_h m\|_M \leq \|m\|_M$, for all $m \in M$. We also denote $p_h : M \backslash N(\pi_h) \to M_h$ the normalized projection form M onto M_h defined by $p_h(m) := \pi_h m / \|\pi_h m\|_M, \quad \forall m \in M,\ \pi_h m \neq 0$. Note that if $\|m\|_M = 1$, $p_h(m)$ satisfies $\|p_h(m) - m\|_M \leq \sqrt{2}\,\|\pi_h m - m\|_M$.*

In the following, we will assume that π_h is also a contraction for the E-norm. That means,

$$\forall m \in E \subset M, \quad \|\pi_h m\|_E \leq \|m\|_E\,. \tag{2.2}$$

This hypothesis is true in the case $E := L^\infty(\Omega)$, $M := L^2(\Omega)$ and $M_h := \mathbb{P}^0(\Omega_h)$ as in Example 2.3.

Definition 2.5. *For any non zero $\mu \in M$, we define its relative error of interpolation onto M_h by*

$$\varepsilon_h^{int}(\mu) := \frac{\|\pi_h \mu - \mu\|_M}{\|\mu\|_M}. \tag{2.3}$$

As the sequence of subspaces $V_h \subset V$ approaches V, we define V_h' the space of all linear forms over V_h endowed with the norm

$$\|\boldsymbol{\varphi}\|_{V_h'} := \sup_{\boldsymbol{v}\in V_h} \frac{\langle \boldsymbol{\varphi}, \boldsymbol{v}\rangle_{V_h', V_h}}{\|\boldsymbol{v}\|_V}.$$

Note that $\boldsymbol{f} \mapsto \boldsymbol{f}|_{V_h}$ defines a natural map from V' onto V_h' and then any $\boldsymbol{f} \in V$ naturally defines a unique element $\boldsymbol{f}|_{V_h}$ of V_h' (and we continue to call it $\boldsymbol{f}$). Then any non zero right-hand side linear form $\boldsymbol{f} \in V'$ is approached by a finite dimensional linear form $\boldsymbol{f}_h \in V_h'$ and we define its relative error of interpolation as follows.

DEFINITION 2.6. *The relative error of interpolation* ε_h^{rhs} *between* $\boldsymbol{f} \neq \boldsymbol{0}$ *and* $\boldsymbol{f}_h$ *is defined by*

$$\varepsilon_h^{rhs} := \frac{\|\boldsymbol{f}_h - \boldsymbol{f}\|_{V_h'}}{\|\boldsymbol{f}\|_{V'}}. \tag{2.4}$$

2.3. Interpolation of the operator. We approach the operator $T \in \mathscr{L}(M, V')$ by a finite dimensional operator $T_h \in \mathscr{L}(M_h, V_h')$. The error of approximation is defined as $T - T_h$ for the $\mathscr{L}(E_h, V_h')$ norm which is weaker than assuming that the distance between T and T_h is small in $\mathscr{L}(M_h, V_h')$. We remind the reader that $E_h := E \cap M_h$ endowed with the E-norm.

DEFINITION 2.7. *The interpolation error* ε_h^{op} *between* T *and* T_h *is defined by*

$$\varepsilon_h^{op} := \|T_h - T\|_{E_h, V_h'} := \sup_{\mu\in E_h} \frac{\|(T_h - T)\mu\|_{V_h'}}{\|\mu\|_E}. \tag{2.5}$$

This error contains both the interpolation error over the approximation spaces and the possible noise in measurements used to build T_h.

Remark 2.8. The reason of the choice of norms comes from the main application where $M := L^2(\Omega)$, $E := L^\infty(\Omega)$, $V := H_0^1(\Omega, \mathbb{R}^d)$ and $T\mu := -\nabla \cdot (\mu S)$ with $S \in L^\infty(\Omega, \mathbb{R}^{d\times d})$. This operator is approached by $T_h\mu := -\nabla \cdot (\mu S_h)$ where S_h is a discrete and possibly noisy version of S. In this case, the interpolation error $S_h - S$ is expected to be small in $L^2(\Omega, \mathbb{R}^{d\times d})$ but not in $L^\infty(\Omega, \mathbb{R}^{d\times d})$. This conduces to small interpolation error $\varepsilon_h^{\mathrm{op}}$ thanks to the control

$$\|(T_h - T)\mu\|_{H^{-1}(\Omega)} \leq \|S_h - S\|_{L^2(\Omega)} \|\mu\|_{L^\infty(\Omega)}, \quad \forall \mu \in M_h. \tag{2.6}$$

but $T_h - T$ has no reason to be small in $\mathscr{L}(M_h, V_h')$ (See example 2.9). This definition of $\varepsilon_h^{\mathrm{op}}$ matches well practical situations like medical imaging for instance where S might be a discontinuous map with *a priori* unknown surfaces of discontinuity. Therefore it makes sense to consider $S_h - S$ small in $L^2(\Omega, \mathbb{R}^{d\times d})$ but not in $L^\infty(\Omega, \mathbb{R}^{d\times d})$. The next example 2.9 below explains this situation in dimension one.

Example 2.9. In dimension one, take $\Omega := (-1, 1)$, $M = L^2(\Omega)$, $E = L^\infty(\Omega)$ and $V = H_0^1(\Omega)$. Take $S \in L^\infty(\Omega)$ and define $T\mu := -(\mu S)'$. Fix $h > 0$ and consider any uniform subdivision $\Omega_h \subset \Omega$ of size h containing the segment $I_h := (-h/2, h/2)$ (hence 0 is not a node). Define the interpolation spaces $M_h := \mathbb{P}^0(\Omega_h)$, $V_h := \mathbb{P}_0^1(\Omega_h)$. Chose $S = 1 + \chi_{(0,1)}$ and $S_h = 1 + \chi_{(\frac{h}{2},1)} \in M_h$ and $T_h\mu := -(\mu S_h)'$. An explicit computation gives

$$\|S_h - S\|_{L^2(\Omega)}^2 = \frac{h}{2} \quad \text{i.e.} \quad \|S_h - S\|_{L^2(\Omega)} = \mathcal{O}\left(\sqrt{h}\right).$$

Thanks to (2.6), we also get that $\|T_h - T\|_{E_h,V_h'} = \mathcal{O}\left(\sqrt{h}\right)$.

Consider now the sequence $\mu_h = h^{-1/2}\chi_{I_h}$ which satisfies $\|\mu_h\|_{L^2(\Omega)} = 1$ and a basis test function $v_h \in V_h$ supported in $[-h/2, 3h/2]$ and such that $v_h(h/2) = 1$. It satisfies $\|v_h\|_{H_0^1(-1,1)} = \sqrt{2/h}$. We can write

$$\langle -(\mu_h(S_h - S))', v_h\rangle_{H^{-1},H_0^1} = \int_{I_h} \mu_h(S_h - S)v_h' = h^{-1/2},$$

hence

$$\sup_{v\in V_h} \frac{\langle -(\mu_h(S_h - S))', v\rangle_{H^{-1},H_0^1}}{\|v\|_{H_0^1(-1,1)}} \geq \frac{\langle -(\mu_h(S_h - S))', v_h\rangle_{H^{-1},H_0^1}}{\|v_h\|_{H_0^1(-1,1)}} = \frac{\sqrt{2}}{2},$$

and then $\|T_h - T\|_{M_h,V_h'} \geq \frac{\sqrt{2}}{2}$. As a consequence $T_h - T$ is not getting small for the $\mathscr{L}(M_h, V_h')$-norm.

3. The generalized *inf-sup* constant.

In this section we generalize the notion of *inf-sup* constant to any operators T in $\mathscr{L}(M, V')$. Let us first define three useful constants for such operators.

Definition 3.1. *For any $T \in \mathscr{L}(M, V')$, we call*

$$\alpha(T) := \inf_{\mu\in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} \quad \textit{and} \quad \rho(T) := \sup_{\mu\in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M}.$$

we also call $\delta(T) := \sqrt{\rho(T)^2 - \alpha(T)^2}$.

We now extend the notion of *inf-sup* constant of the gradient operator to any operators of $\mathscr{L}(M, V')$. As the existence of a null space of dimension one is not guaranteed[1], we first propose this very general definition of the generalized *inf-sup* constant called $\beta(T)$.

3.1. Definition and properties.

Definition 3.2. *The* inf-sup *constant of direction $e \in M$, $e \neq 0$ of the operator $T \in \mathscr{L}(M, V')$ is the non-negative number*

$$\beta_e(T) := \inf_{\substack{\mu\in M\\ \mu\perp e}} \frac{\|T\mu\|_{V'}}{\|\mu\|_M}.$$

The generalized inf-sup *constant of T is now defined by*

$$\beta(T) := \sup_{\substack{e\in M\\ \|e\|_M=1}} \beta_e(T).$$

It is mandatory here to show that this definition indeed extends the classical definition of the *inf-sup* constant known for ∇-type operators (with a null space of dimension one).

[1] Depending on $S(x)$, the operator $T : \mu \mapsto -\nabla\cdot(\mu S)$ may have various type of null spaces. In one hand, in [1] it has been shown that if S is smooth and everywhere invertible, then $N(T) = \{0\}$ if and only if $S^{-1}\nabla\cdot S$ is not a gradient. In the other hand, if S vanishes in a subset $\omega \subset \Omega$, then any function μ supported inside ω belongs to $N(T)$.

Proposition 3.3. *Let $T \in \mathscr{L}(M, V')$ and $z \in M$ such that $\|z\|_M = 1$ and $\|T\,z\|_{V'}^2 \leq \alpha(T)^2 + \varepsilon^2$ for some $\varepsilon \geq 0$. We have*

$$\beta_z(T)^2 \leq \beta(T)^2 \leq \beta_z(T)^2 + \varepsilon(\delta(T) + \varepsilon).$$

In case where $\varepsilon = 0$, it implies that $\beta(T) = \beta_z(T)$.

The proof of this result uses the self-adjoint operator $S_T \in \mathscr{L}(M)$ canonically associated with T.

Lemma 3.4. *For any $T \in \mathscr{L}(M, V')$, there exists $S_T \in \mathscr{L}(M)$ self-adjoint positive semi-definite such that for any $\mu \in M$, $\|T\mu\|_{V'}^2 = \langle S_T\mu, \mu\rangle_M$.*

Proof. Call $\Phi : V' \to V$ the Riesz isometric identification defined by $\langle \Phi f, \boldsymbol{v}\rangle_V = \langle f, \boldsymbol{v}\rangle_{V',V}$ for any $\boldsymbol{f} \in V'$, $v \in V$. Call also $T^* : V \to H$ the adjoint operator of T. We have for any $\mu \in M$,

$$\|T\mu\|_{V'}^2 = \|\Phi T\mu\|_V^2 = \langle T\mu, \Phi T\mu\rangle_{V',V} = \langle \mu, T^*\Phi T\mu\rangle_M = \langle S_T\mu, \mu\rangle_M .$$

where $S_T := T^*\Phi T : M \to M$ is a self-adjoint positive semi-definite operator. ∎

Proof. (of Proposition 3.3) The first inequality comes from the definition of $\beta(T)$. For the second, take $e \in M$ of norm one and consider $m \in E \cap \{z\}^\perp$ of norm one. If $e \perp z$ then $z \in \{e\}^\perp$ and immediately $\beta_e(T)^2 \leq \|T\,z\|_{V'}^2 \leq \alpha(T)^2 + \varepsilon^2 \leq \beta_z(T)^2 + \varepsilon(\delta(T) + \varepsilon)$.

Suppose now that $\langle e, z\rangle_M \neq 0$. Consider $a = -\langle m, e\rangle_M \,/\, \langle z, e\rangle_M$ and $\mu := az + m$. It is clear that $\mu \in \{e\}^\perp$ and $\|\mu\|_M^2 = a^2 + 1$. Using Lemma 3.4, we write

$$\begin{aligned}
\|T\mu\|_{V'}^2 &= \langle S_T\mu, \mu\rangle_M = a^2 \langle S_T z, z\rangle_M + 2a \langle S_T z, m\rangle_M + \langle S_T m, m\rangle_M \\
&= a^2 \|T\,z\|_{V'}^2 + 2a \langle S_T z, m\rangle_M + \|Tm\|_{V'}^2 \\
&\leq (1 + a^2) \|Tm\|_{V'}^2 + a^2\varepsilon^2 + 2|a| \left|\langle S_T z, m\rangle_M\right| .
\end{aligned}$$

Using Proposition A.1 we bound $|\langle S_T z, m\rangle_M|$ by $\varepsilon\delta(T)$ and then

$$\begin{aligned}
\frac{\|T\mu\|_{V'}^2}{\|\mu\|_M^2} &\leq \|Tm\|_{V'}^2 + \varepsilon^2 + \varepsilon\delta(T) \\
\inf_{\substack{\mu \in E \\ \mu \perp e}} \frac{\|T\mu\|_{V'}^2}{\|\mu\|_M^2} &\leq \|Tm\|_{V'}^2 + \varepsilon(\delta(T) + \varepsilon) \\
\beta_e(T)^2 &\leq \|Tm\|_{V'}^2 + \varepsilon(\delta(T) + \varepsilon).
\end{aligned}$$

This last statement is true for any $m \in M \cap \{z\}^\perp$ of norm one so we can take the infimum over m to get $\beta_e(T)^2 \leq \beta_z(T)^2 + \varepsilon(\delta(T) + \varepsilon)$. We conclude now by taking the supremum over e. ∎

As a consequence of Proposition 3.3, the generalized *inf-sup* constant has a simpler formula in the case of an operator with trivial null space.

Corollary 3.5. *If $N(T) \neq \{0\}$, consider any $z \in N(T)$ such that $\|z\|_M = 1$. Then we have $\beta(T) = \beta_z(T)$.*

If $T = \nabla$, the classical definition of $\beta(\nabla)$ given in (1.8) matches the definition 3.2.

Remark 3.6. This corollary leads to an alternative definition of $\beta(T)$ which does not depend on the choice of z in $N(T)$ (even for a dimension greater than one). Moreover, we see that $\beta(T) > 0$ implies $\dim N(T) = 1$. Indeed, if $N(T) > 1$, then there exist $z_1, z_2 \in N(T)$ and $\beta(T) = \beta_{z_1}(T) > 0$ with $z_1 \perp z_2$ such that $T(z_1) = T(z_2) = 0$ and $z_1, z_2 \neq 0$. Moreover, as $\|Tz_2\| \geq \beta_{z_1}(T)\|z_2\|$, we have $z_2 = 0$ which is a contradiction.

It is possible to extend a little this corollary to a class of operators with trivial null space if the infimum value of the operator on the unit sphere is reached.

Remark 3.7. The case $\varepsilon = 0$ in Proposition 3.3 leads to an alternative definition of $\beta(T)$ if the infimum $\alpha(T)$ is reached. Note that this definition does not depend on the choice of z. Moreover the condition $\varepsilon = 0$ is fulfilled in particular if T is a finite rank or finite dimensional operator.

If the infimum value $\alpha(T)$ is not reached on the unit sphere, we keep the general definition 3.2.

3.2. Discrete *inf-sup* constant. The different constants related to the approximated operator $T_h \in \mathscr{L}(M_h, V_h')$ come from the same definition than for the operator $T \in \mathscr{L}(M, V')$. Simply remark that as T_h is a finite dimensional operator, the infimum in

$$\alpha(T_h) := \inf_{\mu \in M_h} \frac{\|T_h \mu\|_{V_h'}}{\|\mu\|_M} \tag{3.1}$$

is reached by a direction $z_h \in M_h$ such that $\|z_h\|_M = 1$. This means that $\|T_h z_h\|_{V_h'} = \alpha(T_h)$. As a consequence, following Corollary **??**, the *inf-sup* constant of T_h is given by

$$\beta(T_h) := \inf_{\substack{\mu \in M_h \\ \mu \perp z_h}} \frac{\|T_h \mu\|_{V_h'}}{\|\mu\|_M}. \tag{3.2}$$

This discrete *inf-sup* constant is the key element to establish the stability of the discrete inverse problem and as we will see, its behaviors when $h \to 0$ will determine the convergence of the solution of the discrete problem to the exact solution. In a similar way than for the classical *inf-sup* constant, the behavior of the discrete $\inf - \sup$ constant $\beta(T_h)$ can be catastrophic in the sense that it can vanish to zero if $h \to 0$. This strongly depends on the choice of interpolation pair of spaces (M_h, V_h). For instance, if the discrete operator $T_h : M_h \to V_h'$ is under-determinated, one may have $\beta(T_h) = 0$. In a same manner than in [8], we give a definition of the discrete *inf-sup* condition.

Definition 3.8. *We say that the sequence of operators $(T_h)_{h>0}$ satisfies the discrete* inf-sup *condition if there exists $\beta^* > 0$ such that*

$$\beta^* \leq \beta(T_h), \quad \forall h > 0. \tag{3.3}$$

Remark 3.9. In this work, we do not prove that the discrete *inf-sup* condition is satisfied by some specific choices of discretized operators $T_h : M_h \to V_h'$. We mention it here as a condition for uniform stability with respect to h, (see Theorems 4.1 4.7). We only aim at giving discrete stability estimates that involves $\beta(T_h)$ for a fixed $h > 0$.

3.3. Upper semi-continuity of the *inf-sup* constant. A legitimate question about the discrete *inf-sup* constant is to know if it can be greater that the continuous *inf-sup* constant if the discretization spaces are well chosen. Inspired by a classical result on the discrete *inf-sup* of the divergence that can be found in [8] for instance, we state and prove in this subsection that the discrete *inf-sup* constant is upper semi-continuous when $h \to 0$. This concludes that the discrete *inf-sup* constant $\beta(T_h)$ is always asymptotically worse than the continuous *inf-sup* constant $\beta(T)$.

THEOREM 3.10 (Upper semi-continuity). *If the operator error ε^{op} defined in (2.5) converges to 0 when $h \to 0$, then*

$$\limsup_{h\to 0} \alpha(T_h) \le \alpha(T).$$

Moreover, if the problem $T\,z = \mathbf{0}$ admits a solution $z \in E$ with $\|z\|_M = 1$ and if the sequence $(T_h)_{h>0}$ satisfies the discrete inf-sup *condition (see Definition 3.8), then*

$$\limsup_{h\to 0} \beta(T_h) \le \beta(T).$$

Remark 3.11. This result is useful to understand that no discretization can get a better stability constant than $\beta(T)$. The question of the convergence of $\alpha(T_h)$ and $\beta(T_h)$ toward respectively $\alpha(T)$ and $\beta(T)$ is not treated here; it is clearly not a simple question. It is already known as a difficult issue concerning *inf-sup* constant of the gradient operator. See [5] for more details about this question.

Remark 3.12. An interesting consequence of this result is that, in case of an operator T with non-trivial null space, the fact that $(T_h)_{h>0}$ satisfies the discrete *inf-sup* condition implies that $\beta(T) > 0$ which means that T has closed range (see [6, p. 47]). It could be used to prove the closed range property for some operators. For instance, to our knowledge, the minimal conditions on $S \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ that make $T : \mu \mapsto -\nabla\cdot(\mu S)$ a closed range operator are not known.

Proof. (of Theorem 3.10) First define the sequence of set

$$C_h := \left\{ \mu \in M_h \mid (\varepsilon_h^{\text{op}})^{1/2} \|\mu\|_E \le \|\mu\|_M \right\}.$$

For any $h > 0$ and $\mu \in C_h$ we get

$$\begin{aligned} \|T_h\mu\|_{V_h'} &\le \|T\mu\|_{V_h'} + \|(T_h - T)\mu\|_{V_h'} \le \|T\mu\|_{V'} + \varepsilon_h^{\text{op}} \|\mu\|_E \\ &\le \|T\mu\|_{V'} + (\varepsilon_h^{\text{op}})^{1/2} \|\mu\|_M . \end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned} \alpha(T_h) &\le \frac{\|T\mu\|_{V'}}{\|\mu\|_M} + (\varepsilon_h^{\text{op}})^{1/2}, \quad \forall \mu \in C_h \\ \alpha(T_h) &\le \inf_{\mu\in C_h} \frac{\|T_h\mu\|_{V_h'}}{\|\mu\|_M} + (\varepsilon_h^{\text{op}})^{1/2}. \end{aligned}$$

This is true for any $h > 0$ so $\limsup_{h\to 0} \alpha(T_h) \le \limsup_{h\to 0} \inf_{\mu\in C_h} \frac{\|T\mu\|_{V'}}{\|\mu\|_M}$.

As proposition B.4 shows that $\lim_{h\to 0} C_h = M$ in the sense of Definition B.1, using that T is continuous over the sphere $\{\mu \in M \mid \|\mu\|_M = 1\}$ we can use Proposition

B.3 that says

$$\limsup_{h\to 0} \inf_{\mu\in C_h} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} \le \inf_{\mu\in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} = \alpha(T)$$

which gives the first result.

For the second result, consider the sequence $(z_h)_{h>0}$ that satisfies $\|z_h\|_M = 1$ and $\|T_h z_h\|_{V_h'} = \alpha(T_h)$. Then $\beta(T_h) = \beta_{z_h}(T_h)$. For any $h > 0$ and $\mu \in C_h \cap \{z_h\}^\perp$, similarly to (3.4), we get

$$\|T_h\mu\|_{V_h'} \le \|T\mu\|_{V'} + (\varepsilon_h^{\text{op}})^{1/2} \|\mu\|_M ,$$

and then by definition of $\beta(T_h)$,

$$\beta(T_h) \le \frac{\|T\mu\|_{V'}}{\|\mu\|_M} + (\varepsilon_h^{\text{op}})^{1/2}, \quad \forall \mu \in C_h \cap \{z_h\}^\perp$$
$$\beta(T_h) \le \inf_{\mu\in C_h\cap\{z_h\}^\perp} \frac{\|T_h\mu\|_{V_h'}}{\|\mu\|_M}.$$

This is true for any $h > 0$ so we deduce

$$\limsup_{h\to 0} \beta(T_h) \le \limsup_{h\to 0} \inf_{\mu\in C_h\cap\{z_h\}^\perp} \frac{\|T\mu\|_{V'}}{\|\mu\|_M}.$$

Now as Theorem 4.1 says that the sequence z_h converges to z in M and Proposition B.5 gives that $\lim_{h\to 0} C_h \cap \{z_h\}^\perp = M \cap \{z\}^\perp$, we can use Proposition B.3 that says

$$\limsup_{h\to 0} \inf_{\mu\in C_h\cap\{z_h\}^\perp} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} \le \inf_{\mu\in M\cap\{z\}^\perp} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} = \beta_z(T) = \beta(T)$$

which gives the second result. ∎

4. Error estimates.

4. Error estimates. In this section, we state and prove the error estimates that are stability estimates for the approximated problem $T_h\mu_h = \boldsymbol{f}_h$.

4.1. Error estimate in the case $\boldsymbol{f} = \mathbf{0}$.

THEOREM 4.1 (Error estimate in the case $\boldsymbol{f} = \mathbf{0}$). *Consider $T \in \mathscr{L}(M, V')$ and let $z \in E$ be a solution of $T z = \mathbf{0}$ with $\|z\|_M = 1$ and assume that h is small enough to have $\varepsilon_h^{int}(z) \le 1/2$. Consider $z_h \in M_h$ a solution of*

$$\|T_h z_h\|_{V_h'} = \alpha(T_h) \quad \textit{with} \quad \|z_h\|_M = 1 \quad \textit{and} \quad \langle z_h, z\rangle_M \ge 0. \tag{4.1}$$

If $\beta(T_h) > 0$ we have

$$\|z_h - p_h(z)\|_M \le \frac{4}{\beta(T_h)}\big(\sqrt{2}\ \|z\|_E\ \varepsilon_h^{op} + 2\rho(T)\varepsilon_h^{int}(z)\big).$$

Where ε_h^{op} and ε_h^{int} are defined in (2.5) and (2.3). Moreover, if $\varepsilon_h^{op} \to 0$ and (T_h) satisfies the discrete inf-sup *condition (3.3), then $\|z_h - z\|_M \to 0$.*

Remark 4.2.

1. Note that if $\varepsilon_h^{\text{op}} \to 0$, since $\alpha(T) = 0$, we have, from Theorem 3.10, that $\alpha(T_h) \to 0$. Moreover, if the discrete inf sup condition (equation (3.3)) is satisfied, then z_h is defined uniquely.

2. It is necessary to assume $z \in E$ to overcome the fact that $T_h - T$ is controlled in $\mathscr{L}(E_h, V_h')$ but not in $\mathscr{L}(M_h, V_h')$. See section 2.3 for more details. In the framework of the inverse elastography problem, the hypothesis $z \in E := L^\infty(\Omega)$ is not restrictive as physical parameters of biological tissues have bounded values with some known *a priori* bounds.
3. The normalized projection $p_h(z)$ of z is the best possible approximation of z in M_h with the constraint of norm one.
4. Problem (4.1) admits a solution z_h as T_h is a finite dimensional operator. The condition $\langle z_h, z\rangle_M \geq 0$ is only here to choose between the two solutions z_h and $-z_h$ and is not of crucial importance.
5. This result provides a quantitative error estimate as $\beta(T_h)$ can be computed from T_h as the second smallest singular value (see Subsection5.1) and all the error terms on the right-hand side can be estimated (at least an upper bound can be given).

Before giving the proof of Theorem 4.1, we first establish and prove a more general result.

PROPOSITION 4.3. *Consider $T_1 \in \mathscr{L}(M, V')$ let $z_1 \in E$ be a solution of*

$$\|T_1\, z_1\|_{V'} \leq \alpha(T_1) + \varepsilon_1 \quad \textit{with} \quad \|z_1\|_M = 1$$

where $\varepsilon_1 \geq 0$. Fix $r \geq \|z_1\|_E$. For any $T_2 \in \mathscr{L}(M, V')$, consider a solution $z_2 \in E$ of

$$\|T_2\, z_2\|_{V'} \leq \alpha(T_2) + \varepsilon_2 \quad \textit{with} \quad \|z_2\|_M = 1 \quad \textit{and} \quad \langle z_1, z_2\rangle_M \geq 0.$$

If $\beta_{z_2}(T_2) > 0$ we have $\|z_2 - z_1\|_M \leq \frac{\sqrt{2}}{\beta_{z_2}(T_2)}\left(2r\,\|T_2 - T_1\|_{E,V'} + 2\alpha(T_1) + 2\varepsilon_1 + \varepsilon_2\right)$ and if $\varepsilon_2 = 0$ this reads $\|z_2 - z_1\|_M \leq \frac{\sqrt{2}}{\beta(T_2)}\left(2r\,\|T_2 - T_1\|_{E,V'} + 2\alpha(T_1) + 2\varepsilon_1\right)$.

Proof. Write $z_1 = tz_2 + m$ where $t \in [0, 1]$ and $m \perp z_2$. We have that $1 = t^2 + \|m\|_M^2$. Then $z_1 - z_2 = (t-1)z_2 + m$ and so $\|z_2 - z_1\|_M^2 = 2(1-t) \leq 2(1-t^2) \leq 2\,\|m\|_M^2$. Then $\|z_2 - z_1\|_M \leq \sqrt{2}\,\|m\|_M$. Now use the definition of $\beta_{z_2}(T_2)$ to write

$$\begin{aligned}\beta_{z_2}(T_2)\,\|m\|_M \leq \|T_2 m\|_{V'} \leq \|T_2 z_1\|_{V'} + \|T_2 z_2\|_{V'} &\leq \|T_2 z_1\|_{V'} + \alpha(T_2) + \varepsilon_2 \\ &\leq 2\,\|T_2 z_1\|_{V'} + \varepsilon_2\end{aligned}$$

and remark that $\|T_2 z_1\|_{V'} \leq \|(T_2 - T_1)z_1\|_{V'} + \|T_1 z_1\|_{V'} \leq r\,\|T_2 - T_1\|_{E,V'} + \|T_1 z_1\|_{V'}$ which implies that

$$\|T_2 z_1\|_{V'} \leq r\,\|T_2 - T_1\|_{E,V'} + \alpha(T_1) + \varepsilon_1.$$

We deduce that $\beta_{z_2}(T_2)\,\|m\|_M \leq 2r\,\|T_2 - T_1\|_{E,V'} + 2\alpha(T_1) + 2\varepsilon_1 + \varepsilon_2$ and then $\|z_2 - z_1\|_M \leq \frac{\sqrt{2}}{\beta_{z_2}(T_2)}\left(2r\,\|T_2 - T_1\|_{E,V'} + 2\alpha(T) + 2\varepsilon_1 + \varepsilon_2\right)$. ∎

We now give the proof of Theorem 4.1:

Proof. First remark that the infimum in (4.1) is reached here because T_h is a finite dimensional operator. Consider $T|_{M_h} : M_h \to V_h'$ and call $g_h := Tp_h(z)$. This quantity is small in V_h' as

$$\begin{aligned}\|\boldsymbol{g}_h\|_{V_h'} = \|Tp_h(z)\|_{V_h'} = \|T(p_h(z) - z)\|_{V_h'} &\leq \|T\|_{M,V'}\,\|p_h(z) - z\|_M \\ &\leq \sqrt{2}\rho(T)\varepsilon_h^{\text{int}}(z).\end{aligned}$$

From this, we deduce that $\alpha(T|_{M_h}) \leq \sqrt{2}\rho(T)\varepsilon_h^{\text{int}}(z)$ and that $p_h(z)$ is solution of

$$\|T|_{M_h} p_h(z)\|_{V_h'} \leq \alpha(T|_{M_h}) + \varepsilon \quad \text{with} \quad \|p_h(z)\|_M = 1,$$

with $\varepsilon = \sqrt{2}\rho(T)\varepsilon_h^{\text{int}}(z)$. Due to Hypothesis (2.2) and $\varepsilon_h^{\text{int}}(z) \leq 1/2$ we have

$$\|p_h(z)\|_E = \frac{\|\pi_h z\|_E}{\|\pi_h z\|_M} \leq 2\frac{\|z\|_E}{\|z\|_M} \leq 2r.$$

Applying now Proposition (4.3) on operators $T_1 = T|_{M_h}$ and $T_2 = T_h$ both in $\mathscr{L}(M_h, V_h')$ with $z_1 = p_h(z)$, $z_2 = z_h$, $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = 0$. We get

$$\begin{aligned}
\|z_h - p_h(z)\|_M &\leq \frac{\sqrt{2}}{\beta(T_h)} \left(4r\, \varepsilon_h^{\text{op}} + 2\alpha(T|_{M_h}) + 2\varepsilon\right) \\
&\leq \frac{\sqrt{2}}{\beta(T_h)} \left(4r\, \varepsilon_h^{\text{op}} + 4\sqrt{2}\rho(T)\varepsilon_h^{\text{int}}(z)\right) \\
&\leq \frac{4}{\beta(T_h)} \left(\sqrt{2}\, r\, \varepsilon_h^{\text{op}} + 2\rho(T)\varepsilon_h^{\text{int}}(z)\right).
\end{aligned}$$

For the convergence, the additional hypothesis give the convergence of the right-hand side. We use that $p_h(z) \to z$ to conclude. ◻

4.2. Error estimates in the case $\boldsymbol{f} \neq \mathbf{0}$. We give and prove a first stability result based on the constant $\alpha(T_h)$.

THEOREM 4.4 (Error estimate using $\alpha(T_h)$). *Consider $\mu \in E$ a solution of $T\mu = \boldsymbol{f}$ with $\boldsymbol{f} \neq 0$ and assume that h is small enough to have $\varepsilon_h^{int}(\mu) \leq 1/2$. Fix $r := \|\mu\|_E / \|\mu\|_M$. Consider now $\mu_h \in M_h$ a solution of $\mu_h = \underset{m \in M_h}{\arg\min} \|T_h m - \boldsymbol{f}_h\|_{V_h'}$. If $\alpha(T_h) > 0$, we have*

$$\frac{\|\mu_h - \pi_h \mu\|_M}{\|\pi_h \mu\|_M} \leq \frac{4}{\alpha(T_h)} \left[r\, \varepsilon_h^{op} + \rho(T)\left(\varepsilon_h^{rhs} + \varepsilon_h^{int}(\mu)\right)\right].$$

Where ε_h^{op}, ε_h^{rhs} and ε_h^{int} are defined in (2.5), (2.4) and (2.3). Moreover, if there exists $\alpha^* > 0$ such that $\alpha(T_h) \geq \alpha^*$ for all $h > 0$ and if $\varepsilon_h^{op} \to 0$ and $\varepsilon_h^{rhs} \to 0$ when $h \to 0$, then $\|\mu_h - \mu\|_M \to 0$ when $h \to 0$.

Remark 4.5. Note that if $\alpha(T_h) > 0$ for all $h > 0$, then μ_h is uniquely defined and moreover $\varepsilon_h^{\text{op}} \to 0$ and if $\alpha(T_h) \geq \alpha_* > 0$, Theorem 3.10 assures that $\alpha(T) \geq \alpha^* > 0$ which guarantee the uniqueness of μ.

Remark 4.6. This result makes sense in practice even if $\alpha(T_h)$ goes to zero. Indeed, at a fixed $h > 0$, $\alpha(T_h)$ can be computed from T_h as the first singular value and all the error terms on the right-hand side can be estimated (at least an upper bound can be given). It then gives a quantitative error bound on the reconstruction that can be useful no matter with the asymptotic behavior of $\alpha(T_h)$.

Proof. First note that from the hypothesis $\varepsilon_h^{\text{int}}(\mu) \leq 1/2$ we have that $\|\mu\|_M \leq 2\|\pi_h \mu\|_M$ and $\|\pi_h \mu\|_E \leq \|\mu\|_E \leq r\|\mu\|_M \leq 2r\|\pi_h \mu\|_M$and $\|\boldsymbol{f}\|_{V'} \leq \rho(T)\|\mu\|_M$.

From the definition of $\alpha(T_h)$ we write

$$\begin{aligned}
\alpha(T_h)\,\|\mu_h - \pi_h\mu\|_M &\le \|T_h\mu_h - T_h\pi_h\mu\|_{V_h'} \le \|T_h\mu_h - \boldsymbol{f}_h\|_{V_h'} + \|T_h\pi_h\mu - \boldsymbol{f}_h\|_{V_h'} \\
&\le 2\,\|T_h\pi_h\mu - \boldsymbol{f}_h\|_{V_h'} \\
&\le 2\,\|T\mu - \boldsymbol{f}_h\|_{V_h'} + 2\,\|T\,\pi_h\mu - T\mu\|_{V_h'} + 2\,\|(T_h - T)\,\pi_h\mu\|_{V_h'} \\
&\le 2\,\|\boldsymbol{f} - \boldsymbol{f}_h\|_{V_h'} + 2\rho(T)\,\|\pi_h\mu - \mu\|_M + 2\,\varepsilon_h^{\mathrm{op}}\,\|\pi_h\mu\|_E \\
&\le 2\varepsilon_h^{\mathrm{rhs}}\,\|\boldsymbol{f}\|_{V'} + 2\rho(T)\varepsilon_h^{\mathrm{int}}(\mu)\,\|\mu\|_M + 4r\,\varepsilon_h^{\mathrm{op}}\,\|\pi_h\mu\|_M \\
&\le 2\rho(T)\left(\varepsilon_h^{\mathrm{rhs}} + \varepsilon_h^{\mathrm{int}}(\mu)\right)\|\mu\|_M + 4r\,\varepsilon_h^{\mathrm{op}}\,\|\pi_h\mu\|_M \\
&\le 4\left[\rho(T)\left(\varepsilon_h^{\mathrm{rhs}} + \varepsilon_h^{\mathrm{int}}(\mu)\right) + r\varepsilon_h^{\mathrm{op}}\right]\|\pi_h\mu\|_M\,.
\end{aligned}$$

∎

We now state and prove the main stability estimate concerning the general problem $T\mu = \boldsymbol{f}$ with a non zero right-hand side. This result uses $\beta(T_h)$ which is always better than $\alpha(T_h)$. The price of this change is that the stability estimates only holds in the hyperplane $\{z_h\}^\perp$, where z_h is the vector that minimizes $\|T_hz_h\|_{V_h'}$ on the unit sphere.

Theorem 4.7 (Error estimate using $\beta(T_h)$). *Consider $\mu \in E$ a solution of $T\mu = \boldsymbol{f}$ with $\boldsymbol{f} \neq 0$ and assume that h is small enough to have $\varepsilon_h^{int}(\mu) \le 1/2$. Fix $r := \|\mu\|_E \,/\, \|\mu\|_M$. Consider $z_h \in M_h$ a solution of*

$$\|T_hz_h\|_{V_h'} = \alpha(T_h) \quad \textit{with} \quad \|z_h\|_M = 1.$$

Consider now $\mu_h \in M_h$ a solution of

$$\mu_h = \operatorname*{arg\,min}_{\substack{m\in M_h\\ m\perp z_h}} \|T_hm - \boldsymbol{f}_h\|_{V_h'}\,, \quad \textit{with } \mu_h \perp z_h. \tag{4.2}$$

If $\beta(T_h) > 0$, there exists $t \in \mathbb{R}$ such that $\mu_{h,t} := tz_h + \mu_h$ satisfies

$$\frac{\|\mu_{h,t} - \pi_h\mu\|_M}{\|\pi_h\mu\|_M} \le \frac{4}{\beta\,(T_h)}\left[r\,\varepsilon_h^{op} + \rho(T)\left(\varepsilon_h^{rhs} + \varepsilon_h^{int}(\mu)\right) + \frac{\alpha\,(T_h)}{2}\right].$$

Where ε_h^{op}, ε_h^{rhs} and ε_h^{int} are defined in (2.5), (2.4) and (2.3).

Remark 4.8. This result has to be used as soon as Theorem 4.4 is irrelevant because $\alpha(T_h)$ is too small. It somehow kills the degenerated direction z_h and gives a possibly better estimate for the computed solution up to an unknown component in the direction z_h.

Remark 4.9. This result gives also the algorithmic procedure to approach the exact solution μ:

1. Identify z_h with stability thanks to Theorem 4.1.
2. Solve the problem (4.2) to identify μ_h.
3. Find the best approximation $tz_h + \mu_h$ by choosing a correct coefficient $t \in \mathbb{R}$ using any additional scalar information on the exact solution such as its mean, its background value, a punctual value, etc…

Remark 4.10. This result provides a quantitative error estimate as $\alpha(T_h)$ and $\beta(T_h)$ can be computed from T_h as the two first singular values and all the error terms on the right-hand side can be estimated (at least an upper bound can be given).

Before giving the proof of this Theorem, let us state and prove an intermediate result.

Proposition 4.11. *Consider* $T_1 \in \mathscr{L}(M,V')$ $\boldsymbol{f}_1 \in V'$, $\boldsymbol{f}_1 \neq 0$ *and let* $\mu_1 \in E$ *be a solution of* $T_1\,\mu_1 = \boldsymbol{f}_1$. *Fix* $r := \|\mu_1\|_E \,/\, \|\mu_1\|_M$ *and for any* $T_2 \in \mathscr{L}(M,V')$, *consider a solution* $z_2 \in E$ *of*

$$\|T_2\, z_2\|_{V'} \leq \alpha(T_2) + \varepsilon_2 \quad \textit{and} \quad \|z_2\|_M = 1$$

and consider a solution $\mu_2 \in E$ *of*

$$T_2\,\mu_2 = \boldsymbol{f}_2 \quad \textit{and} \quad \mu_2 \perp z_2.$$

If $\beta_{z_2}(T_2) > 0$, *there exists* $t \in \mathbb{R}$ *such that* $\mu_{2,t} := tz_2 + \mu_2$ *satisfies*

$$\frac{\|\mu_{2,t} - \mu_1\|_M}{\|\mu_1\|_M} \leq \frac{1}{\beta_{z_2}(T_2)} \left(\frac{\|\boldsymbol{f}_2 - \boldsymbol{f}_1\|_{V'}}{\|\mu_1\|_M} + r\,\|T_2 - T_1\|_{E,V'} + \alpha(T_2) + \varepsilon_2 \right).$$

Moreover, if $\varepsilon_2 = 0$ *it reads*

$$\frac{\|\mu_{2,t} - \mu_1\|_M}{\|\mu_1\|_M} \leq \frac{1}{\beta(T_2)} \left(\frac{\|\boldsymbol{f}_2 - \boldsymbol{f}_1\|_{V'}}{\|\mu_1\|_M} + r\,\|T_2 - T_1\|_{E,V'} + \alpha(T_2) \right).$$

Proof. Denote $\mu_{2,t} := tz_2 + \mu_2$ with $t := \langle \mu_1, z_2 \rangle_M$. With this choice, we have that $(\mu_{2,t} - \mu_1) \perp z_2$. From the definition of $\beta_{z_2}(T_2)$, we write

$$\begin{aligned}
\beta_{z_2}(T_2)\,\|\mu_{2,t} - \mu_1\|_M &\leq \|T_2\,\mu_{2,t} - T_2\,\mu_1\|_{V'} \\
&\leq \|T_2\mu_2 - T_1\mu_1\|_{V'} + |t|\,\|T_2\,z_2\|_{V'} + \|(T_2 - T_1)\mu_1\|_{V'} \\
&\leq \|\boldsymbol{f}_2 - \boldsymbol{f}_1\|_{V'} + \|\mu_1\|_M\,(\alpha(T_2) + \varepsilon_2) + \|T_2 - T_1\|_{E,V'}\,\|\mu_1\|_E\,. \\
&\leq \|\boldsymbol{f}_2 - \boldsymbol{f}_1\|_{V'} + \|\mu_1\|_M \left(\alpha(T_2) + \varepsilon_2 + r\,\|T_2 - T_1\|_{E,V'} \right).
\end{aligned}$$

∎

We can now give the proof of Theorem 4.7.

Proof. (of Theorem 4.7) Consider $T|_{M_h} : E_h \to V_h'$ and call $\boldsymbol{g}_h := T\pi_h\mu$. Remark that $\|\pi_h\mu\|_E \leq \|\mu\|_E \leq r\,\|\mu\|_M \leq 2r\,\|\pi_h\mu\|_M$. Applying Proposition 4.11 to the operators $T_1 := T|_{M_h}$, $T_2 := T_h$ both in $\mathscr{L}(M_h, V_h')$, with $\boldsymbol{f}_1 := \boldsymbol{g}_h$, $\boldsymbol{f}_2 := T_h\mu_h$ both in V_h' and with $\mu_1 := \pi_h\mu$, $\mu_2 := \mu_h$. We get the existence of $t \in \mathbb{R}$ such that

$$\frac{\|\mu_{h,t} - \pi_h\mu\|_M}{\|\pi_h\mu\|_M} \leq \frac{1}{\beta(T_h)} \left(\frac{\|T_h\mu_h - \boldsymbol{g}_h\|_{V_h'}}{\|\pi_h\mu\|_M} + 2r\,\varepsilon_h^{\mathrm{op}} + \alpha(T_h) \right).$$

Now we bound $\|T_h\mu_h - \boldsymbol{g}_h\|_{V_h'}$ as follows:

$$\|T_h\mu_h - \boldsymbol{g}_h\|_{V_h'} \leq \|T_h\mu_h - \boldsymbol{f}_h\|_{V_h'} + \|\boldsymbol{g}_h - \boldsymbol{f}_h\|_{V_h'}\,.$$

To deal with the first term, we define $p := \pi_h\mu - \langle \pi_h\mu, z_h \rangle_M\, z_h$ orthogonal to z_h. We have

$$\begin{aligned}
\|T_h\mu_h - \boldsymbol{f}_h\|_{V_h'} \leq \|T_hp - \boldsymbol{f}_h\|_{V_h'} &\leq \|T_h\pi_h\mu - \boldsymbol{f}_h\|_{V_h'} + \|T_hz_h\|_{V_h'}\,\|\pi_h\mu\|_M \\
&\leq \|T\pi_h\mu - \boldsymbol{f}_h\|_{V_h'} + \|(T_h - T)\pi_h\mu\|_{V_h'} + \alpha(T_h)\,\|\pi_h\mu\|_M \\
&\leq \|\boldsymbol{g}_h - \boldsymbol{f}_h\|_{V_h'} + \varepsilon_h^{\mathrm{op}}\,\|\pi_h\mu\|_E + \alpha(T_h)\,\|\pi_h\mu\|_M \\
&\leq \|\boldsymbol{g}_h - \boldsymbol{f}_h\|_{V_h'} + (2r\,\varepsilon_h^{\mathrm{op}} + \alpha(T_h))\,\|\pi_h\mu\|_M\,.
\end{aligned}$$

Now the second term is bounded as follows:

$$\begin{aligned}\|\boldsymbol{g}_h - \boldsymbol{f}_h\|_{V_h'} &\leq \|\boldsymbol{g}_h - \boldsymbol{f}\|_{V_h'} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{V_h'} \leq \|T\pi_h\mu - T\mu\|_{V_h'} + \varepsilon_h^{\text{rhs}} \|\boldsymbol{f}\|_{V'} \\ &\leq \rho(T)\varepsilon_h^{\text{int}}(\mu) \|\mu\|_M + \rho(T)\varepsilon_h^{\text{rhs}} \|\mu\|_M \leq \rho(T) \|\mu\|_M \left(\varepsilon_h^{\text{int}}(\mu) + \varepsilon_h^{\text{rhs}}\right) \\ &\leq 2\rho(T) \|\pi_h\mu\|_M \left(\varepsilon_h^{\text{int}}(\mu) + \varepsilon_h^{\text{rhs}}\right).\end{aligned}$$

This last line is true because the hypothesis $\varepsilon_h^{\text{int}}(\mu) \leq 1/2$ implies that $\|\mu\|_M \leq 2\|\pi_h\mu\|_M$. Putting things together, it come that

$$\frac{\|T_h\mu_h - \boldsymbol{g}_h\|_{V_h'}}{\|\pi_h\mu\|_M} \leq 4\rho(T)\left(\varepsilon_h^{\text{int}}(\mu) + \varepsilon_h^{\text{rhs}}\right) + 2r\,\varepsilon_h^{\text{op}} + \alpha(T_h)$$

and then

$$\frac{\|\mu_{h,t} - \pi_h\mu\|_M}{\|\pi_h\mu\|_M} \leq \frac{2}{\beta(T_h)}\left[2\rho(T)\left(\varepsilon_h^{\text{int}}(\mu) + \varepsilon_h^{\text{rhs}}\right) + 2r\,\varepsilon_h^{\text{op}} + \alpha(T_h)\right].$$

□

5. Numerical results.

In this section we provide numerical applications of Theorems 4.1 and 4.7 and we present the general methodology to numerically approach the solution of the equation (1.1) in various contexts. In the whole section, we stay in the framework where $M := L^2(\Omega)$, $E := L^\infty(\Omega)$ and $V := H_0^1(\Omega, \mathbb{R}^d)$.

In subsection 5.2, we exhibit a simple and efficient pair of approximation spaces (M_h, V_h) called the honeycomb discretization pair, that numerically satisfies the discrete *inf-sup* condition.

For all the numerical experiments, we use the Matlab environment with some elements of the PDE toolbox. We first determine the matrix $\mathcal{M}$ and then, the determination of the constants α and β and the determination of the solution of the homogeneous problem is done using the singular values decomposition method (svds in Matlab). The determination of the solution for the heterogeneous problem is simply done using the classical linear system solver (mldivide in Matlab). For the high degree finite element spaces ($\mathbb{P}^2, \mathbb{P}^3, \mathbb{P}^4$), we use the getfem (see [19]) environnement on Matlab to generate the matrix $\mathcal{M}$.

5.1. Matrix formulation of the discretized problem.

In this section, we describe the matrix formulation of the discrete problem (1.5) which gives a way to use the stability theorems in practice. Let us fix a discretization size $h > 0$ and pick a pair of finite dimensional subspaces $M_h \subset M$ and $V_h \subset V$. Let $(\varepsilon_1, \dots, \varepsilon_n)$ be a basis of M_h and let $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_p)$ be a basis of V_h. We define $\mathcal{T} \in \mathbb{R}^{p\times n}$ and $\boldsymbol{b} \in \mathbb{R}^p$ the matrix versions of the discrete operator T_h and the right-hand side $\boldsymbol{f}_h$as the matrices

$$\mathcal{T}_{ij} := \langle T_h\varepsilon_j, \boldsymbol{e}_i\rangle_{V_h', V_h}, \quad \text{and} \quad \boldsymbol{b}_i := \langle \boldsymbol{f}_h, \boldsymbol{e}_i\rangle_{V_h', V_h}. \tag{5.1}$$

As no ambiguity can occur, we adopt the notation for $\mu := \sum_j \mu_j\varepsilon_j \in M_h$ and $\boldsymbol{\mu} := (\mu_1, \dots \mu_n)^T$ and the same notation for $\boldsymbol{v} := \sum_i v_i\boldsymbol{e}_i \in V_h$ and $\boldsymbol{v} = (v_1, \dots, v_p)^T \in \mathbb{R}^p$. We have the correspondence

$$\langle T_h\mu, v\rangle_{V_h', V_h} = \boldsymbol{v}^T\mathcal{T}\boldsymbol{\mu}.$$

We now call $(\mathcal{S}_M)_{ij} := \langle\varepsilon_i, \varepsilon_j\rangle_M$ and $(\mathcal{S}_V)_{ij} := \langle\boldsymbol{e}_i, \boldsymbol{e}_j\rangle_V$. They enable to compute the norm in M and V through the formulas $\|\mu\|_M^2 = \sum_{i,j}\mu_i\mu_j\langle\varepsilon_i, \varepsilon_j\rangle_M = \boldsymbol{\mu}^T\mathcal{S}_M\boldsymbol{\mu}$, and $\|\boldsymbol{v}\|_V^2 = \sum_{i,j} v_i v_j \langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle_V = \boldsymbol{v}^T \mathcal{S}_V \boldsymbol{v}$. If we denote $\mathcal{B}_M$ and $\mathcal{B}_V$ the square root matrices of $\mathcal{S}_M$ and $\mathcal{S}_V$ (i.e. such that $\mathcal{B}_M^2 = \mathcal{S}_M$), we have that $\|\mu\|_M = \|\mathcal{B}_M \boldsymbol{\mu}\|_2$ and $\|\boldsymbol{v}\|_V = \|\mathcal{B}_V \boldsymbol{v}\|_2$. Hence the constant $\alpha(T_h)$ is given by

$$\begin{aligned} \alpha(T_h) &= \inf_{\boldsymbol{\mu}\in\mathbb{R}^n} \sup_{\boldsymbol{v}\in\mathbb{R}^p} \frac{\boldsymbol{v}^T \mathcal{T} \boldsymbol{\mu}}{\|\mathcal{B}_M \boldsymbol{\mu}\|_2 \|\mathcal{B}_V \boldsymbol{v}\|_2} \\ &= \inf_{\boldsymbol{\mu}\in\mathbb{R}^n} \sup_{\boldsymbol{v}\in\mathbb{R}^p} \frac{\boldsymbol{v}^T \mathcal{B}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1} \boldsymbol{\mu}}{\|\boldsymbol{\mu}\|_2 \|\boldsymbol{v}\|_2} = \inf_{\boldsymbol{\mu}\in\mathbb{R}^n} \frac{\left\|\mathcal{B}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1} \boldsymbol{\mu}\right\|_2}{\|\boldsymbol{\mu}\|_2}. \end{aligned} \tag{5.2}$$

which is the smallest singular value of the matrix

$$\mathcal{M} := \mathcal{B}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1}$$

or also the square root of the smallest eigenvalue of $\mathcal{M}^T \mathcal{M} = \mathcal{B}_M^{-1} \mathcal{T}^T \mathcal{S}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1}$.

Call now $\boldsymbol{z} \in \mathbb{R}^n$ the first singular vector of $\mathcal{M}$ (hence associated with $\alpha(T_h)$) or the first eigenvector of $\mathcal{M}^T \mathcal{M}$. It is equal to the solution $z_h := \sum_j z_j \varepsilon_j \in M_h$ of (4.1) up to a change of sign.

Remark 5.1. The basis matrices $\mathcal{B}_M$ and $\mathcal{B}_V$ are mandatory to get the exact solution $\alpha(T_h)$ and z_h as defined in (4.1). As $\alpha(T_h)$ is expected to be small, it is possible to consider directly the first singular vector of the matrix $\mathcal{T}$ itself. The numerical computation gets a bit simpler but creates an additional error which is not controlled by the theory described herein.

We can now compute the discrete *inf-sup* constant of T_h:

$$\begin{aligned} \beta(T_h) &= \inf_{\substack{\boldsymbol{\mu}\in\mathbb{R}^n \\ \boldsymbol{\mu}\perp \mathcal{S}_M \boldsymbol{z}}} \sup_{\boldsymbol{v}\in\mathbb{R}^p} \frac{\boldsymbol{v}^T \mathcal{T} \boldsymbol{\mu}}{\|\mathcal{B}_M \boldsymbol{\mu}\|_2 \|\mathcal{B}_V \boldsymbol{v}\|_2} \\ &= \inf_{\substack{\boldsymbol{\mu}\in\mathbb{R}^n \\ \boldsymbol{\mu}\perp \boldsymbol{z}}} \sup_{\boldsymbol{v}\in\mathbb{R}^p} \frac{\boldsymbol{v}^T (\mathcal{B}_V^{-1})^T \mathcal{T} \mathcal{B}_M^{-1} \boldsymbol{\mu}}{\|\boldsymbol{\mu}\|_2 \|\boldsymbol{v}\|_2} = \inf_{\substack{\boldsymbol{\mu}\in\mathbb{R}^n \\ \boldsymbol{\mu}\perp \boldsymbol{z}}} \frac{\|\mathcal{M}\boldsymbol{\mu}\|_2}{\|\boldsymbol{\mu}\|_2} \end{aligned} \tag{5.3}$$

which is the second smallest singular value of the matrix $\mathcal{M}$ or also the square root of the second smallest eigenvalue of $\mathcal{M}^T \mathcal{M}$. Finally, in order to give the solution of (4.2) in Theorem 4.7, we rewrite the problem under a matrix formulation:

$$\begin{aligned} \min_{\substack{m\in M_h \\ m\perp z_h}} \|T_h m - \boldsymbol{f}_h\|_{V_h'} &= \min_{\substack{m\in M_h \\ m\perp z_h}} \sup_{\boldsymbol{v}\in V_h} \frac{\langle T_h m - \boldsymbol{f}_h, \boldsymbol{v}\rangle_{V_h', V_h}}{\|\boldsymbol{v}\|_V} = \min_{\substack{\boldsymbol{\mu}\in\mathbb{R}^n \\ \boldsymbol{\mu}\perp \boldsymbol{z}}} \sup_{\boldsymbol{v}\in\mathbb{R}^p} \frac{\boldsymbol{v}^T (\mathcal{T}\boldsymbol{\mu} - \boldsymbol{b})}{\|\mathcal{B}_V \boldsymbol{v}\|_2} \\ &= \min_{\substack{\boldsymbol{\mu}\in\mathbb{R}^n \\ \boldsymbol{\mu}\perp \boldsymbol{z}}} \sup_{\boldsymbol{v}\in\mathbb{R}^p} \frac{\boldsymbol{v}^T \mathcal{B}_V^{-1} (\mathcal{T}\boldsymbol{\mu} - \boldsymbol{b})}{\|\boldsymbol{v}\|_2} = \min_{\substack{\boldsymbol{\mu}\in\mathbb{R}^n \\ \boldsymbol{\mu}\perp \boldsymbol{z}}} \left\|\mathcal{B}_V^{-1} (\mathcal{T}\boldsymbol{\mu} - \boldsymbol{b})\right\|_2. \end{aligned}$$

Call now $\widetilde{\mathcal{T}} := \begin{bmatrix} \mathcal{T} \\ \boldsymbol{z}^T \end{bmatrix}$, $\widetilde{\boldsymbol{b}} := \begin{bmatrix} \boldsymbol{b} \\ 0 \end{bmatrix}$ and $\widetilde{\mathcal{B}}_V := \begin{bmatrix} \mathcal{B}_V & 0 \\ 0 & 1 \end{bmatrix}$ we aim at solving

$$\widetilde{\mathcal{B}}_V^{-1} \widetilde{\mathcal{T}} \boldsymbol{\mu} = \widetilde{\mathcal{B}}_V^{-1} \widetilde{\boldsymbol{b}}$$

in the sense of least squares which is equivalent to define $\boldsymbol{\mu} := (\widetilde{\mathcal{T}}^T \widetilde{\mathcal{S}}_V^{-1} \widetilde{\mathcal{T}})^{-1} \widetilde{\mathcal{T}}^T \widetilde{\mathcal{S}}_V^{-1} \widetilde{\boldsymbol{b}}$.

5.2. The honeycomb pair of finite element spaces. After numerous tests with various finite element pair of spaces, it appears that a specific pair of spaces gather a large amount of advantages for the specific use in our inverse parameter

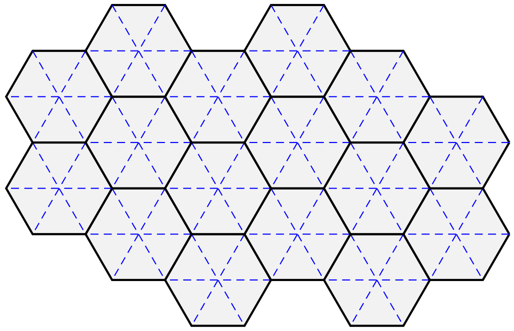

FIG. 1. *Honeycomb space discretization. In plain black, the hexagonal subdivision and in dashed blue, the triangular subdivision.*

problem. This pair (M_h, V_h) is the so called honeycomb discretization pair. Like in Figure 1, define a regular hexagonal subdivision of Ω denoted $\{\Omega_{h,j}^{\mathrm{hex}}\}_{j=1,\dots,N_h^{\mathrm{hex}}}$ where $h > 0$ is the edge length of the hexagons and N_h^{hex} is the number of hexagons used. We then call $\Omega_h \subset \Omega$ the subdomain defined by this subdivision. That means

$$\overline{\Omega_h} = \bigcup_{j=1}^{N_h^{\mathrm{hex}}} \overline{\Omega_{h,j}^{\mathrm{hex}}}.$$

Now we consider the uniform triangular sub-mesh defined by subdividing each hexagon in six equilateral triangles of size h. This subdivision is denoted $\{\Omega_{h,k}^{\mathrm{tri}}\}_{k=1,\dots,N_h^{\mathrm{tri}}}$ where $N_h^{\mathrm{tri}} := 6N_h^{\mathrm{hex}}$. It is represented in dashed blue in figure 1.

We now define the finite dimensional discretization space M_h of M as the collection of functions $\mu \in L^2(\Omega_h)$ that are constant in each hexagon. In other terms,

$$M_h := \mathbb{P}^0\left(\Omega_h^{\mathrm{hex}}\right) = \left\{\mu \in L^2(\Omega_h) \mid \forall j \ \mu|_{\Omega_{h,j}^{\mathrm{hex}}} \text{ is constant}\right\}.$$

Functions in M_h can be extended by 0 out of Ω_h to get $M_h \subset M$. For the discretization space of V, we chose the classical finite element class $\mathbb{P}_0^1$ over the triangulation. It is made of all the functions of $H_0^1(\Omega_h)$ that are linear over all the triangles. In other terms,

$$V_h := \mathbb{P}_0^1\left(\Omega_h^{\mathrm{tri}}, \mathbb{R}^2\right) = \left\{\boldsymbol{v} \in H_0^1(\Omega_h, \mathbb{R}^2) \mid \forall k \ \boldsymbol{v}|_{\Omega_{h,k}^{\mathrm{tri}}} \text{ is linear}\right\}.$$

Functions in V_h can be extended by $\mathbf{0}$ out of Ω_h to get $V_h \subset V$.

Remark 5.2. This particular choice of finite element spaces gathers several advantages to compare to other more classical pairs:

1. The space $\mathbb{P}^0\left(\Omega_h^{\mathrm{hex}}\right)$ is suitable for discontinuous functions interpolation. This is important as we aim at recovering discontinuous mechanical parameters of biological tissues for instance.
2. The hexagonal discretization of Ω is optimal among the other regular plane tilings (triangle and square) in the sense that it minimizes the ratio of the number of elements N_h^{hex} over the size h. As we see in all the numerical tests, it also provides the smallest error on the reconstruction among all the other pairs of spaces that we have tried. As a conjecture, we believe that this pair

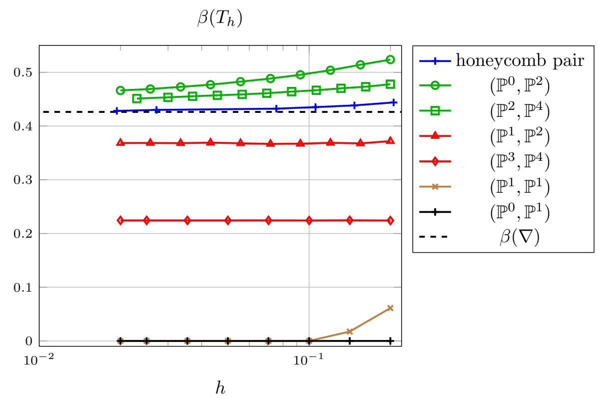

FIG. 2. *Behavior of the discrete* inf-sup *constant* $\beta(T_h)$ *for the inverse gradient problem in the unit square* $\Omega := (0,1)^2$*, for various choices of pair of discretization spaces. The dashed line represents the conjectured value of the* inf-sup *constant* $\beta(\nabla) = \sqrt{1/2 - 1/\pi}$ *of the gradient operator in* Ω.

of spaces should be optimal, in term of error estimate, for a large class of operators T.

3. From a given hexagonal mesh and triangular sub-mesh, spaces $\mathbb{P}^0\left(\Omega_h^{\text{hex}}\right)$ and $\mathbb{P}_0^1\left(\Omega_h^{\text{tri}}, \mathbb{R}^2\right)$ are easy to build from the most classical pair of finite element spaces $\left(\mathbb{P}^0\left(\Omega_h^{\text{tri}}\right), \mathbb{P}^1\left(\Omega_h^{\text{tri}}\right)\right)$.
4. The system of equations $T_h\mu_h = \boldsymbol{f}_h$ is (most of the time) over-determinate as it involves around $2N_h^{\text{hex}}$ equations for N_h^{hex} unknown. Note that as we solve the problem in the sense of least squares, over-determination is not a problem while under-determination is.
5. This pair gives an excellent evaluation of the discrete *inf-sup* constant $\beta(T_h)$ in our numerical exemples that is a key element for discrete stability.

5.3. Inverse gradient problem. Let Ω be the unit square $(0,1)^2$. We approach here the solution $\mu \in L^\infty(\Omega)$ of the problem $-\nabla\mu = \boldsymbol{f}$ where $\boldsymbol{f}$ is given vectorial function. This case correspond to (1.1) where $S = I$ everywhere. In this case, many simplification occur as $T_h := -\nabla|_{M_h}$ and then $\varepsilon_h^{\text{op}} = 0$. Moreover $\rho(T) \leq 1$. In the absence of noise, the result of Theorem 4.7 reads : $\frac{\|\mu_h - \pi_h\mu\|_M}{\|\pi_h\mu\|_M} \leq \frac{4}{\beta(T_h)}\left(\varepsilon_h^{\text{rhs}} + \varepsilon_h^{\text{int}}(\mu)\right)$ where μ_h is the solution of $\min_{\mu\in M_h} \|T_h\mu - \boldsymbol{f}\|_{V_h'}$ under the condition $\mu_h \in L_0^2(\Omega_h)$ i.e. $\int_{\Omega_h} \mu_h = 0$.

Let first compute $\beta(T_h)$ using (5.3) at check its behavior when h go to 0. In figure 2 we see that it seem to converge to some $\beta_0 > 0$ lower than the conjectured *inf-sup* constant $\beta(\nabla) = \sqrt{1/2 - 1/\pi}$ in the unit square (see [8, Theorem 3.3] for details about this conjectured value).

Consider now a smooth map $\mu_1(x) := \cos(10x_1) + \cos(10x_2)$ for $x \in \Omega$, for such a smooth function we expect an error of interpolation in M_h of order $\varepsilon_h^{\text{int}}(\mu_1) = \mathcal{O}(h)$ and an error of interpolation of its gradient on V_h' of order $\varepsilon_h^{\text{rhs}} = \mathcal{O}(h^2)$. Hence the relative error $E_1(h) := \|\mu_{1,h} - \pi_h\mu_1\|_M / \|\pi_h\mu_1\|_M$ is expected to be at least of order $\mathcal{O}(h)$. In figure 4 we observe a convergence of order 2 in absence of noise. We retry the same test with piecewise constant μ_2. Its derivative is approached first in $\mathbb{P}^0(\Omega_h^{\text{tri}})$ to deduce its vectorial form in V_h'. We observe a convergence of order 1/2 in absence of noise.

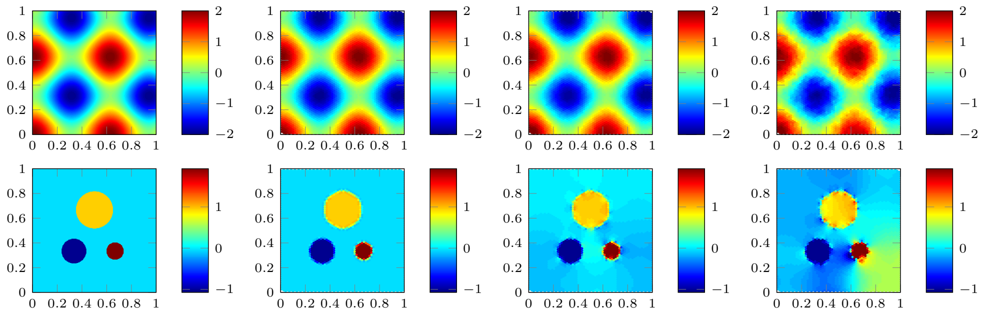

FIG. 3. *Numerical stability of the reconstruction of maps μ_1 and μ_2 using method given by Theorem 4.7 with resolution $h = 0.01$. From left to right: column 1: exact map to recover, 2. reconstruction with no noise, column 3: reconstruction with noise level $\sigma = 1$, column 4: reconstruction with noise level $\sigma = 2$.*

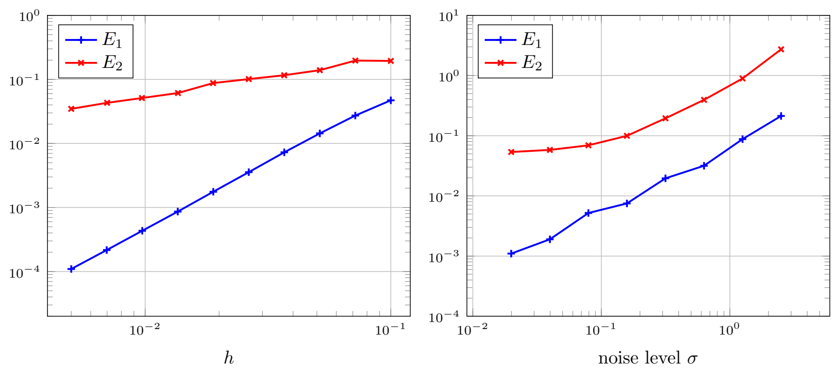

FIG. 4. *Left : relative L^2-error on the reconstruction with respect to h in the absence of noise. Right : relative L^2-error on the reconstruction with respect to the noise level σ with $h = 0.01$.*

To illustrate the stability with respect to noise on the right-had side, we corrupt the data $-\nabla\mu$ with the multiplication term-by-term by $1 + \sigma\mathcal{N}$ where $\sigma > 0$ is the noise level and $\mathcal{N}$ is a Gaussian random variable of variance one.

5.4. Quasi-static elastography.

Forward problem. To illustrate the ability of solving a quasi-static elastography problem in the case $\lambda = 0$ from a single measurement, we compute a virtual data field by solving the linear elastic forward problem

$$\text{(5.4)} \qquad \begin{cases} -\nabla\cdot(2\mu_{\text{exact}}\,\mathcal{E}(\boldsymbol{u})) = \boldsymbol{0} & \text{in } (0,1)^2, \\ 2\mu_{\text{exact}}\,\mathcal{E}(\boldsymbol{u})\cdot\boldsymbol{\nu} = \boldsymbol{g} & \text{on } (0,1)\times\{1\}, \\ \mathcal{E}(\boldsymbol{u})\cdot\boldsymbol{\nu} = \boldsymbol{0} & \text{on } (0,1)\times\{0\}, \\ \boldsymbol{u} = \boldsymbol{0}, & \text{on } \{0,1\}\times(0,1). \end{cases}$$

where μ_{exact} is described in Figure 5. We chose here a constant boundary force $\boldsymbol{g} := (1,-1)^T$. This problem is solved using classical $\mathbb{P}^1$ finite element method over an unstructured triangular mesh. The computed data field $\boldsymbol{u}$ is then stored in a cartesian grid to avoid any numerical inverse crime. It is represented in Figure 5.

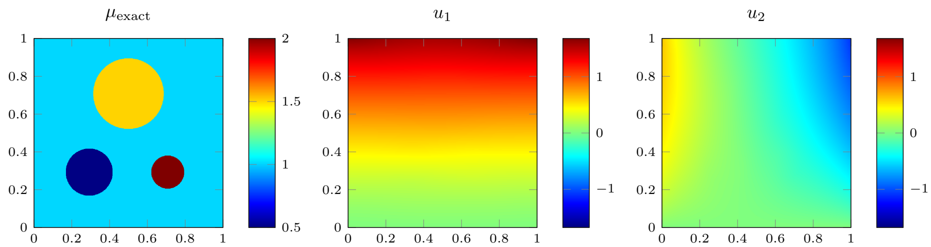

FIG. 5. *First line, from left to right: The exact map μ_{exact}, the two components of the data field $\boldsymbol{u} = (u_1, u_2)$ computed via (5.4), the only data used to inverse the problem.*

Note that μ_{exact} is chosen bounded and piecewise constant, and thanks to the classical elliptic regularity theory the exact strain tensor $S := \mathcal{E}(\boldsymbol{u})$ is piecewise smooth and bounded in Ω (see [17]). Hence, both μ_{exact} and S satisfy the hypotheses used in the error estimates.

Inverse problem. We first choose a pair of approximation spaces (M_h, V_h) and we interpolate the displacement field $\boldsymbol{u}$ on the corresponding mesh nodes as a continuous and piecewise linear map. From this interpolated data, we compute its approached derivative $\nabla \boldsymbol{u}$ by computing the exact piecewise constant derivative of the interpolated displacement field. We deduce the approximation of the strain tensor $S := 2\mathcal{E}(\boldsymbol{u})$ as a piecewise constant map. We then construct the matrix form of the approached operator T_h by formula (5.1). Before applying Theorem 4.1 we compute the discrete values of $\alpha(T_h)$ and $\beta(T_h)$ for few pairs of spaces (see Figure 6). We here control that $\beta(T_h)$ does not vanish and that the ratio $\alpha(T_h)/\beta(T_h)$ is small enough. We recall that this is needed for good error estimates using Theorem 4.1. Note that the honeycomb pair shows a much better behavior than the other consider pairs of spaces. In the results, we denote by "honeycomb pair" the pair of spaces defined in subsection 5.2 and we denote $\mathbb{P}^k$ the classical space of Lagrangian finite element space over an unstructured triangulation (see [12] for precise definitions).

We plot now solutions μ_h of the numerical inversion with various choices of pair of spaces in Figure 7. Then in Figure 8 we present tables of comparisons of different pair of spaces in terms of relative error and complexity through the number of degrees of freedom and number of equations. In particular,

- As expected and for all choice of pair of spaces satisfying inf-sup condition, the numerical approximation $\mathbf{u}$ gives some nice reconstruction of the elastic coefficient $2\mu_{exact}$. Moreover, in each case, we also clearly observe a convergence as $h \to 0$.
- The numerical solutions obtained with the honeycomb approach give some better reconstruction than using other pair of spaces. It can be explained by a better ratio $\alpha(T_h)/\beta(T_h)$.
- The use of high degree as with the pair of spaces ($\mathbb{P}^4$,$\mathbb{P}^2$) raises some numerical memory issues in the computation the matrix $\mathcal{B}_M^{-1}$ and $\mathcal{S}_V^{-1}$. In particular, we don't succeed to reach time steps h smaller than $h = 0.025$ with a standard laptop.
- From a computation cost point of view, the honeycomb approach has also many advantages. The matrix $\mathcal{S}_M$ and $\mathcal{S}_V$ are respectively diagonal and tridiagonal which greatly facilitate the computation of $\mathcal{B}_M^{-1} = \sqrt{\mathcal{S}_M^{-1}}$ and $\mathcal{S}_V^{-1}$.

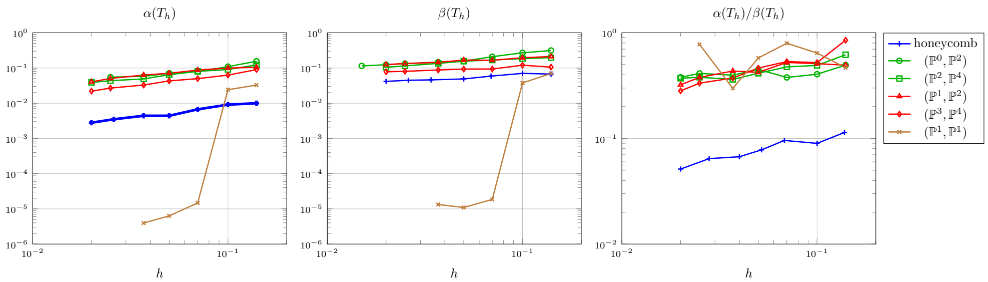

FIG. 6. *Behavior of the contants $\alpha(T_h)$, $\beta(T_h)$ and the ratio $\alpha(T_h)/\beta(T_h)$ for the inverse static elastography problem in the unit square $\Omega := (0,1)^2$, for various choices of pair of discretization spaces.*

Finally, we can reach much finer resolutions than using other finite element space proposed in this paper.

- For all the tested pairs, the matrix is over-determinated and the measured algebraic rank is equal to n. However as the the first singular value is very small to compare to the others, the matrix rank should be considered "numerically speaking equal to $n-1$".

6. Concluding remarks. In this article we have proved the numerical stability of the Galerkin approximation of the inverse parameter problem arising from the elastography in medical imaging. It has been done through a direct discretization of the Reverse Weak Formulation without boundary conditions. The obtained stability estimates arise from a generalization of the *inf-sup* constant (continuous and discrete) to a large class of first order differential operator. These results shed light on the importance of the choice of finite element spaces to assure uniqueness and stability. Various numerical applications have been presented which illustrate the stability theorems. A new pair of finite element spaces based on a hexagonal tilling has been introduced. It showed excellent stability behavior for the specific purpose of this inverse problem.

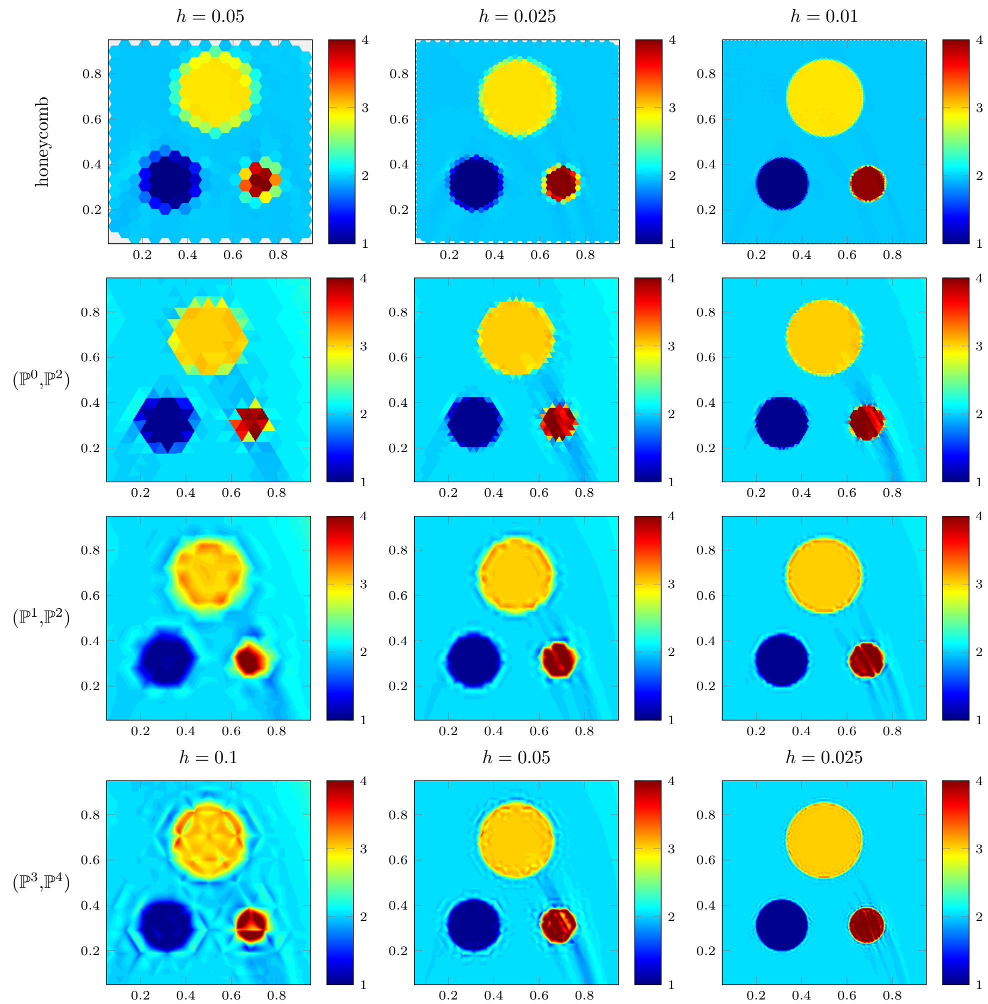

FIG. 7. *Reconstruction of the shear modulus map μ using various pairs of finite element spaces in the subdomain of interest* $(0.1, 0.9)^2$.

$h = 0.05$	E	n	p
honeycomb	9.2%	338	1888
$(\mathbb{P}^0,\mathbb{P}^2)$	9.1%	735	2788
$(\mathbb{P}^1,\mathbb{P}^2)$	8.5%	407	2800
$(\mathbb{P}^3,\mathbb{P}^4)$	5.4%	3424	11k

$h = 0.025$	E	n	p
honeycomb	6.3%	1510	8765
$(\mathbb{P}^0,\mathbb{P}^2)$	6.7%	2982	12k
$(\mathbb{P}^1,\mathbb{P}^2)$	5.7%	1570	12k
$(\mathbb{P}^3,\mathbb{P}^4)$	3.4%	13654	47k

FIG. 8. *Comparison of four pairs of finite element spaces in term of relative error E of the reconstruction, degrees of freedom n, and number of equations p. The product $n\,p$ is an indication of the algorithmic complexity.*

Appendix A. A result on self-adjoint operators.

LEMMA A.1. *Let H be an Hilbert space and $S : H \to H$ be a self-adjoint positive semi-definite linear operator. Call $\alpha^2 := \inf\{\langle Sx, x\rangle_H \mid \|x\|_H = 1\}$ and $z \in H$ such that $\|z\|_H = 1$ and take $\langle Sz, z\rangle_H \le \alpha^2 + \varepsilon^2$ with $\varepsilon > 0$. For any $p \perp z$ with $\|p\|_H = 1$*

we have

$$|\langle Sz,p\rangle_H| \le \varepsilon\sqrt{\rho^2-\alpha^2}$$

x *where* $\rho^2 := \sup\{\langle Sx,x\rangle_H \mid \|x\|_H = 1\}$.

Proof. Consider $t \in (0,1)$, $u_t := -\text{sign}\,\langle Sz,p\rangle_H \sqrt{1-t^2}$ and $z_t := t\,z + u_t\,p$ of norm one. By definition of α we have

$$\begin{aligned}\alpha^2 \le \langle Sz_t,z_t\rangle_H &= t^2\,\langle Sz,z\rangle_H + 2t\,u_t\,\langle Sz,p\rangle_H + u_t^2\,\langle Sp,p\rangle_H\\ &\le t^2(\alpha^2+\varepsilon^2) + 2t\,u_t\,\langle Sz,p\rangle_H + u_t^2\rho^2.\end{aligned}$$

Then

$$\begin{aligned}-2t\,u_t\,\langle Sz,p\rangle_H &\le (t^2-1)\alpha^2 + t^2\varepsilon^2 + u_t^2\rho^2\\ 2t\,|u_t|\,|\langle Sz,p\rangle_H| &\le t^2\varepsilon^2 + u_t^2(\rho^2-\alpha^2)\\ 2\,|\langle Sz,p\rangle_H| &\le \frac{t}{|u_t|}\varepsilon^2 + \frac{|u_t|}{t}(\rho^2-\alpha^2).\end{aligned}$$

This statement is true for any $t \in (0,1)$ so for any $\tau \in (0,1)$ we have

$$2\,|\langle Sz,p\rangle_H| \le \tau\varepsilon^2 + \frac{1}{\tau}(\rho^2-\alpha^2).$$

The minimum of the right-hand side is reached for $\tau = \sqrt{(\rho^2-\alpha^2)/\varepsilon^2}$ which implies that $2\,|\langle Sz,p\rangle_H| \le 2\sqrt{\varepsilon^2(\rho^2-\alpha^2)}$. ∎

Appendix B. Limit of subsets and infimum. Let M be a Hilbert space and let $E \subset M$ be Banach space dense in M. Let $(M_h)_{h>0}$ be a sequence of subspace of E endowed with the M-norm. We assume that the orthogonal projection $\pi_h : M \to M_h$ satisfies

$$\forall x \in E, \quad \|\pi_h x\|_E \le \|x\|_E\,.$$

Definition B.1. *For any sequence* $(A_h)_{h>0}$ *of subsets of* M*, we define its limit as*

$$\lim_{h\to 0} A_h := \left\{x \in M \,|\, \exists (x_h)_{h>0} \subset M,\ \lim_{h\to 0}\|x_h - x\|_M = 0,\ \forall h>0\ x_h \in A_h\right\}.$$

Proposition B.2. $\lim_{h\to 0} A_h$ *is a closed subset of* M *and, if* $A_h \subset X \subset M$ *for all* $h>0$*, then* $\lim_{h\to 0} A_h \subset \overline{X}$.

Proof. Call $A := \lim_{h\to 0} A_h$ and take $x \in \overline{A}$. There exists a sequence $(x_n)_{n\in\mathbb{N}}$ of A such that $\|x - x_n\|_M \le 1/(2n)$ for all $n \in \mathbb{N}^*$. For all $n \in \mathbb{N}^*$, there exists a sequence $(x_n^h)_{h>0}$ such that $\lim_{h\to 0}\left\|x_n^h - x_n\right\|_M = 0$ and $x_n^h \in A_h$ for all $h>0$. Hence there exists $h_n > 0$ such that for all $h \le h_n$ we have $\left\|x_n^h - x_n\right\|_M \le 1/(2n)$. We can decrease h_n to satisfy $h_n < h_{n-1}$ for all $n \ge 2$. Now define the sequence $(y_h)_{h>0}$ as follows: If $h > h_1$, y_h is any element of A_h. If $h \in [h_{n+1}, h_n)$, we take $y_h = x_n^h$. It is clear that $y_h \in A_h$ for all $h > 0$. Moreover, for any $h \le h_n$, $\|y_h - x_n\|_M \le 1/(2n)$ and $\|x - x_n\|_M \le 1/(2n)$ which give $\|y_h - x\|_M \le 1/n$. This shows that $\lim_{h\to 0}\|y_h - x\|_M = 0$ and therefore $x \in A$. The second part of the statement is trivial.

Proposition B.3. *Assume that* $A := \lim_{h\to 0} A_h$ *is not empty and consider a fonction* $f : M \to \mathbb{R}$. *If there exists a subset* $B \subset A$ *such that* f *is continuous in* B *and* $\inf_A f = \inf_B f$ *then we have*

$$\limsup_{h\to 0} \inf_{A_h} f \le \inf_A f.$$

Proof. Take $x \in B$. As $x \in A$, there exists $(x_h)_{h>0}$ such that $x_h \in A_h$ for all $h > 0$ and $\lim_{h\to 0} x_h = x$. For any $h > 0$, $f(x_h) \le f(x) + |f(x_h) - f(x)|$ and $\inf_{A_h} f \le f(x) + |f(x_h) - f(x)|$. Taking the superior limit when $h \to 0$ it comes from the continuity of f at x, $\limsup_{h\to 0} \inf_{A_h} f \le f(x)$ which if true for any $x \in B$ so $\limsup\limits_{h\to 0} \inf\limits_{A_h} f \le \inf\limits_B f = \inf\limits_A f$. □

We assume now that the sequence (M_h) satisfies $\lim_{h\to 0} M_h = M$. We consider a sequence of positive real number α_h that converges zero and a corresponding sequence of subsets $C_h := \{x \in M_h \,|\, \alpha_h \|x\|_E \le \|x\|_M\}$.

Proposition B.4. *The following limit holds:* $\lim\limits_{h\to 0} C_h = M$.

Proof. We prove that $E \subset C := \lim_{h\to 0} C_h$. Take $x \in E\backslash\{0\}$, for h small enough it satisfies $2\alpha_h \|x\|_E \le \|x\|_M$. Consider now its orthogonal projection $\pi_h x$ of x onto M_h. It satisfies $\lim_{h\to 0} \pi_h x = x$. For h small enough $\|x\|_M \le 2\|\pi_h x\|_M$ and then

$$\alpha_h \|\pi_h x\|_E \le \alpha_h \|x\|_E \le \frac{1}{2}\|x\|_M \le \|\pi_h x\|_M$$

which means that $\pi_h x \in C_h$. As a consequence, $x \in \lim_{h\to 0} C_h$. □

Proposition B.5. *Let* $(z_h)_{h>0}$ *be sequence of* M *such that* $\|z_h\|_M = 1$ *and which converges weakly to* $z \ne 0$. *Then*

$$\lim_{h\to 0} \left(C_h \cap \{z_h\}^\perp\right) = M \cap \{z\}^\perp.$$

Proof. Take $x \in \lim_{h\to 0}\left(C_h \cap \{z_h\}^\perp\right)$. There exists (x_h) such that $x_h \in C_h$ and $x_h \perp z_h$ and $x_h \to x$. We have $\langle x, z\rangle_M = \lim_{h\to 0} \langle x, z_h\rangle_M = \lim_{h\to 0} \langle x - x_h, z_h\rangle_M = 0$. Then $x \in M \cap \{z\}^\perp$.

Reversely, take $x \in M \cap \{z\}^\perp$, and fix $\varepsilon > 0$. There exists $x_\varepsilon \in E\backslash\{0\}$ such that $\|x_\varepsilon - x\|_M \le \varepsilon$ and $x_\varepsilon \perp z$ and Consider now the orthogonal projection $\pi_h x_\varepsilon$ of x_ε onto M_h. It satisfies $\lim_{h\to 0} \pi_h x_\varepsilon = x_\varepsilon$. For h small enough $\|x_\varepsilon\|_M \le 2\|\pi_h x_\varepsilon\|_M$. Consider now $\widetilde{z} \in E$ such that $\langle z, \widetilde{z}\rangle_M \ge 1/2$ and $\|\widetilde{z}\|_M = 1$. We define now

$$x_\varepsilon^h = \pi_h x_\varepsilon + \beta_h \pi_h \widetilde{z} \quad \in M_h,$$

with $\beta_h = -\langle \pi_h x_\varepsilon, z_h\rangle_M / \langle \pi_h \widetilde{z}, z_h\rangle_M$ in order to have $x_\varepsilon^h \perp z_h$ for all h. Remark that β_h is wel defined for h small enough as $\langle \pi_h \widetilde{z}, z_h\rangle_M$ converges to $\langle z, \widetilde{z}\rangle_M$ and converges to zero as $\langle \pi_h x_\varepsilon, z_h\rangle_M = \langle x_\varepsilon, z_h\rangle_M + \langle \pi_h x_\varepsilon - x_\varepsilon, z_h\rangle_M$ converges to $\langle x_\varepsilon, z\rangle_M = 0$. Then $x_\varepsilon^h \to x_\varepsilon$. Now we write

$$\left\|x_\varepsilon^h\right\|_E \le \|\pi_h x_\varepsilon\|_E + \beta_h \|\pi_h \widetilde{z}\|_E \le \|x_\varepsilon\|_E + \beta_h \|\widetilde{z}\|_E\,,$$

and $\left\|x_\varepsilon^h\right\|_M \to \|x_\varepsilon\|_M \ne 0$. As a consequence, for h small enough, $\alpha_h \left\|x_\varepsilon^h\right\|_E \le \left\|x_\varepsilon^h\right\|_M$ which means that $x_\varepsilon^h \in C_h \cap \{z_h\}^\perp$ for h small enough. This shows that $x_\varepsilon \in \lim_{h\to 0}\left(C_h \cap \{z_h\}^\perp\right)$. This is true for any $\varepsilon > 0$ and as the limit set is closed, $x \in \lim_{h\to 0}\left(C_h \cap \{z_h\}^\perp\right)$. □

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